Bounding the identifying code number of a graph using its degree parameters

(a probabilistic approach)

Florent Foucaud (LaBRI)

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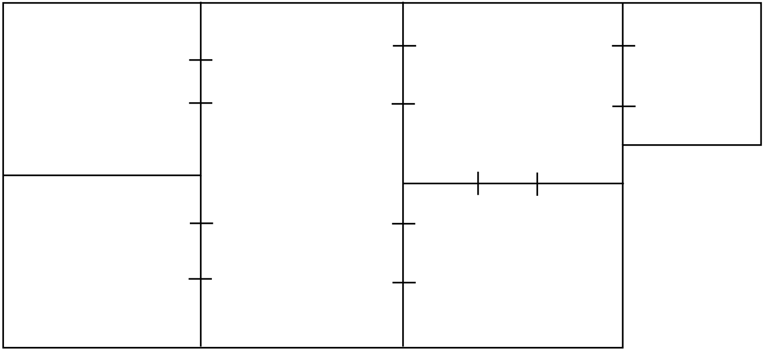
joint work with **Guillem Perarnau** (UPC, Barcelona)

ANR IDEA

Locating a burglar in a museum



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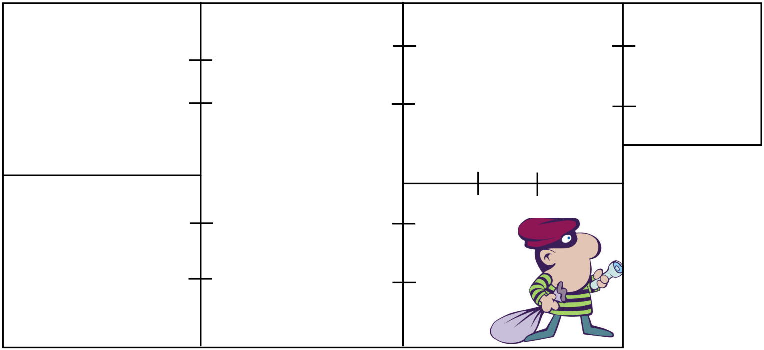

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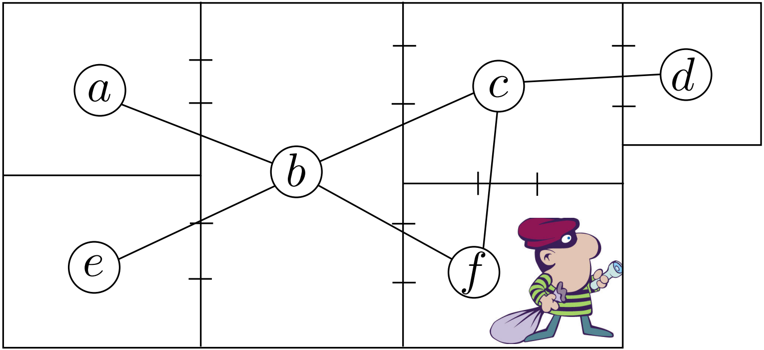
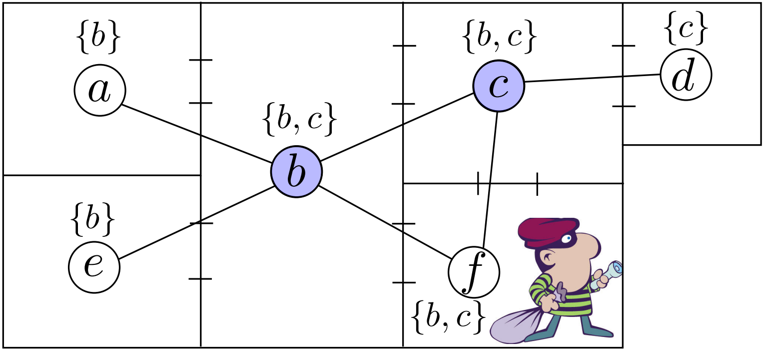


Graph $G = (V, E)$. V: vertices (rooms), $E \subseteq V \times V$: edges (doors)



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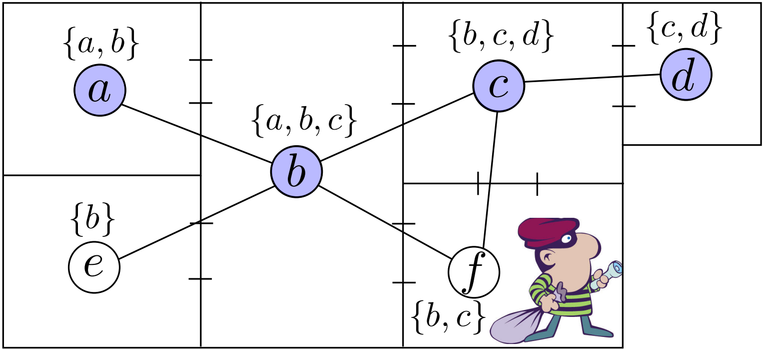

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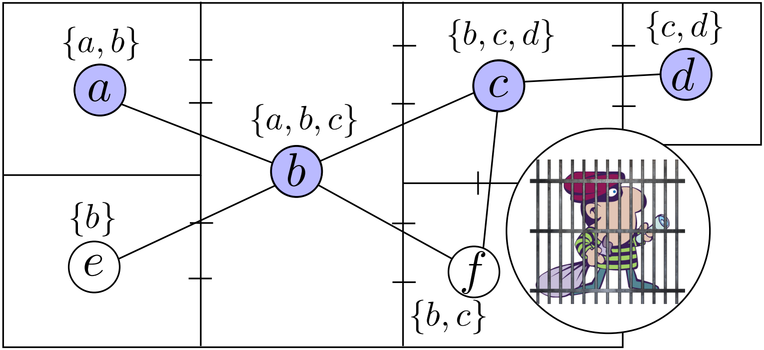

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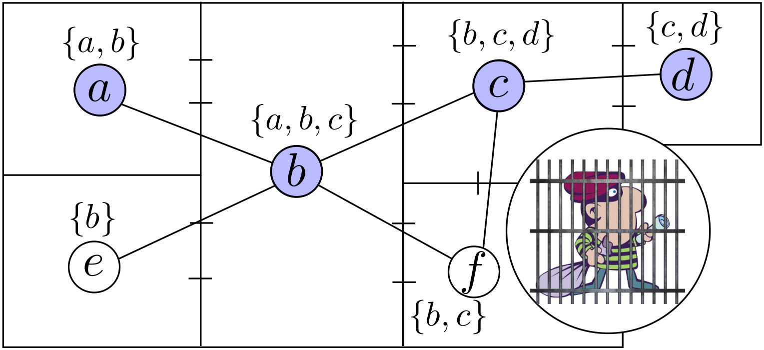

How many detectors do we need?

Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a dominating set in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating code in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Remark

Not all graphs have an identifying code!

Twins $=$ pair u, v such that $N[u] = N[v]$.

A graph is identifiable iff it is twin-free (i.e. it has no twins).

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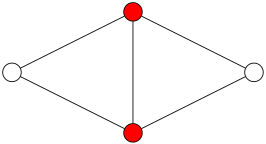

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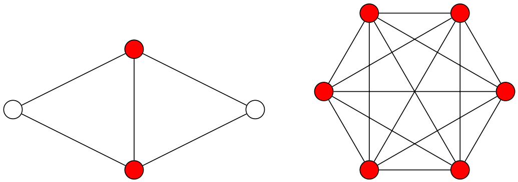
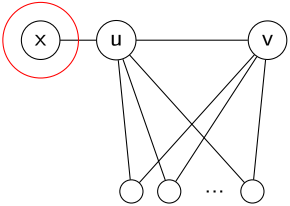


u, v such that $N[v] \Delta N[u] = \{x\}$

Then $x \in C$, forced by uv.



Notation

Let $f(G)$ be the proportion of non forced vertices of G

$$f(G) = \frac{\text{\#non-forced vertices in G}}{\text{\#vertices in G}}$$

Degree parameters of a graph

Graph $G = (V, E)$, vertex $v \in V$.

- degree of v: number of edges it is incident to
- maximum degree d of G: max. degree of a vertex in G
- d-regular graph: all vertices have degree d

Previous results (1/2)

Theorem (Karpovsky, Chakrabarty, Levitin, 1998 + Gravier, Moncel, 2007)

Let G be an identifiable graph with at least one edge, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

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Let G be an identifiable graph with maximum degree d, then

$$\frac{2n}{d+2} \leq \gamma^{\text{ID}}(G)$$

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Conjecture (F., Klasing, Kosowski, Raspaud, 2009+)

Let G be a connected nontrivial identifiable graph of max. degree d. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1)$$

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This would be tight. True for $d=2$ and $d=n-1$.

Previous results (2/2)

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Theorem (F., Guerrini, Kovse, Naserasr, Parreau, Valicov, 2011)

Let G be a connected identifiable graph of maximum degree d. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^5)}$$

If G is d-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$

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If G is d-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$

Theorem (F., Klasing, Kosowski, Raspaud, 2009+)

Let G be a connected identifiable triangle-free graph of max. degree d. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d(1+o_d(1))}$$

The probabilistic method

Technique initiated, among others, by Pál Erdős
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Classic reference: Noga Alon and Joel Spencer, *The probabilistic method*

Upper bounds for $\gamma^{\text{ID}}(G)$

Notation

Let $f(G)$ be the proportion of non forced vertices of G

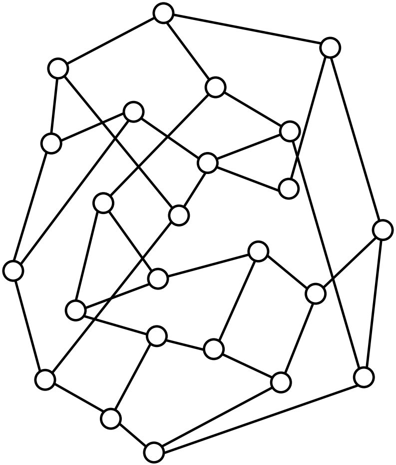
$$f(G) = \frac{\#\text{non-forced vertices in G}}{\#\text{vertices in G}}$$

Theorem (F., Perarnau, 2011+)

There exists an integer d_0 such that for each identifiable graph G on n vertices having maximum degree $d \geq d_0$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot f(G)^2}{85d}$$

Proof - select a set at random...



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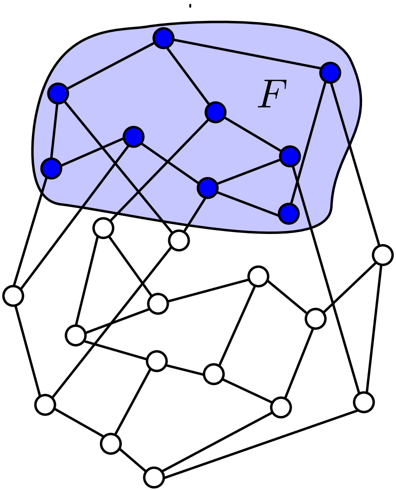

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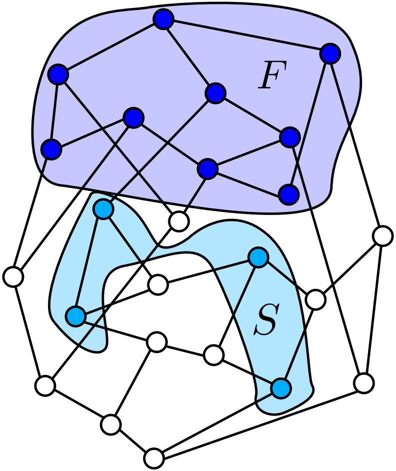
- Select a random set S from $V' = V \setminus F$: each vertex $v \in S$ with prob. p.



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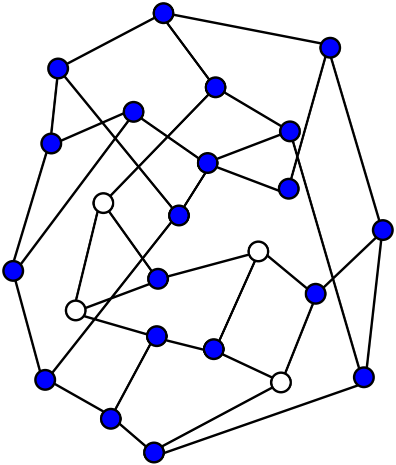
- $\mathcal{C} = V \setminus S$



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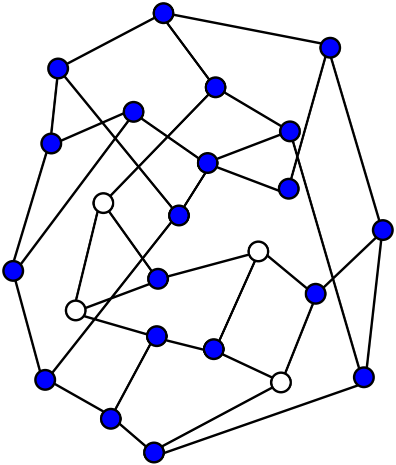

Theorem (Weighted Local Lemma: particular case of the Local Lemma Erdős, Lovász, 1973 - Molloy, Reed, 2001[1])

Let $0 < p \leq \frac{1}{4}$ and $\mathcal{E} = \{E_1, \ldots, E_M\}$ be a set of "bad" events such that each E_i is mutually independent of $\mathcal{E} \setminus (\mathcal{D}_i \cup \{E_i\})$ where $\mathcal{D}_i \subseteq \mathcal{E}$, and

- $Pr(E_i) \leq p^{t_i}$
- $\sum_{E_j \in D_i} (2p)^{t_j} \leq \frac{t_i}{2}$

Then $Pr(\bigcap_{i=1}^{M} \overline{E_i}) \geq \prod_{i=1}^{M} (1 - (2p)^{t_i}) \geq \exp\left\{-2 \log 2 \sum_{i=1}^{m} (2p)^{t_i}\right\} > 0$.

1: Molloy and Reed - *Graph colouring and the probabilistic method*, 2001

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$\Longrightarrow$ If the dependencies are "rare":

with non-zero probability none of the bad events occur

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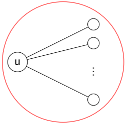
Set the bad events...



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Event A_u

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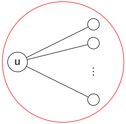

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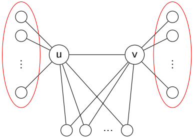

Event $B_{u,v}$

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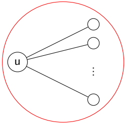

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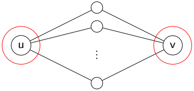

Event $C_{u,v}$

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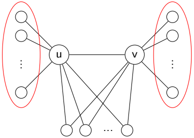

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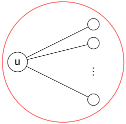

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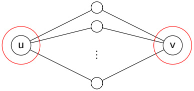

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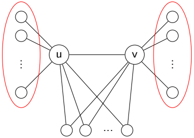

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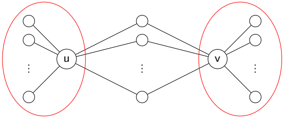

Event $D_{u,v}$

$p^{2d} \leq Pr(D_{u,v}) \leq p^4$

Proof (regular case) - Lovász Local Lemma, technicalities

Theorem (Weighted Local Lemma)

Bad events: $\mathcal{E} = \{E_1, \dots, E_M\}$.

- $Pr(E_i) \leq p^{t_i}$
- $\sum_{E_j \in D_i} (2p)^{t_j} \leq \frac{t_i}{2}$

Then $Pr(\bigcap_{i=1}^{M} \overline{E_i}) \geq \prod_{i=1}^{M} (1 - (2p)^{t_i}) > 0.$

Build the event-intersection table:

	A	B	C	D
A	d^2	$d^3 - 2d^2 + 2d$	$d^4 + d^2 - 2d$	$d^2 - d + 1$
B	$2(d^2 - d + 1)$	$2d^3 - 4d^2 + 4d - 1$	$2d(d^3 - 3d^2 + 4d - 2)$	$d^2 - d$
C	$2d^2 + 2$	$2d(d^2 - 2d + 2)$	$2d^4 - 6d^3 + 9d^2 - 5d - 1$	$d^2 + d - 2$
D	$d + 2$	$d^2 + d$	$d^3 - d$	$2d - 4$

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For an A-event, we need:

$$d^2(2p)^{d+1} + (d^3 - 2d^2 + 2d)(2p)^2 + (d^4 + d^2 - 2d)(2p)^3 + (d^2 - d + 1)(2p)^2 \leq \frac{2}{2} = 1$$

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Taking $p = \frac{1}{kd} \implies$ LLL can be applied

Proof (regular case) - the set can be small...

By the LLL we know that

There exists some set S *with* $\mathbb{E}(|S|) = \frac{nf(G)}{k \cdot d}$ *such that no bad event occurs*

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But by the LLL we know more:

$$\Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > \exp\left\{-2\log 2 \sum_{i=1}^{m} (2p)^{t_i}\right\}$$

Proof (regular case) - the set can be small...

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And we also know: if $S = \emptyset$, $\mathcal{C} = V \setminus S = V$ is a trivial code!

But by the LLL we know more:

$$\Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > \exp\left\{-\frac{9}{k^2 d} n\right\}$$

The probability to have a good set S is at least $\exp\left\{-\frac{9}{k^2 d} n\right\}$

Proof (regular case) - concentration inequality

Theorem (Chernoff bound)

Let $X_1, \ldots, X_m$ a set of i.i.d random variables s.t. $Pr(X_i = 1) = p$ and $Pr(X_i = 0) = 1 - p$ and $X = \sum X_i$. Then

$$\Pr(\mathbb{E}(X) - X > \alpha) \leq \exp\left\{-\frac{\alpha^2}{2mp}\right\}$$

Proof (regular case) - concentration inequality

Theorem (Chernoff bound)

Let $X_1, \ldots, X_m$ a set of i.i.d random variables s.t. $Pr(X_i = 1) = p$ and $Pr(X_i = 0) = 1 - p$ and $X = \sum X_i$. Then

$$\Pr(\mathbb{E}(X) - X > \alpha) \leq \exp\left\{-\frac{\alpha^2}{2mp}\right\}$$

For each $v_i \in V \setminus F$ define the random variable:

$$X_i = \begin{cases} 1 & \text{if } v_i \in C \\ 0 & \text{otherwise} \end{cases}$$

Then, we set $\alpha = \frac{nf(G)}{cd}$. Using $mp = \frac{nf(G)}{kd}$:

$$\Pr\left(\mathbb{E}(X) - X > \frac{nf(G)}{cd}\right) \leq \exp\left\{\frac{kf(G)}{2c^2d}n\right\}$$

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Probability that S is too small: at most $\exp\left\{-\frac{kf(G)}{2c^2d}n\right\}$

Proof (regular case) - size of the code

$$\Pr(S \text{ good}) - \Pr(S \text{ too small}) > 0$$

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$$|\mathcal{C}| = |V \setminus S| \leq n - \frac{nf(G)^2}{85d}$$

Bounding the number of forced vertices

Proposition

Let $f(G)$ be the proportion of non forced vertices of G. Then

$$\frac{1}{d+1} \leq f(G) \leq 1$$

This result is tight for a graph of max. degree $d = n - 1$.

Lemma Bertrand, Hudry, 2005

Let G be an identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex y in $N[x]$.

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Lemma Bertrand, Hudry, 2005

Let G be an identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex y in $N[x]$.

Corollary

The set S of non-forced vertices forms a dominating set. Hence $|S| \geq \frac{n}{d+1}$.

clique number of G: max. size of a complete subgraph in G

Proposition

Let G be a graph of clique number at most k. There exists a function c such that:

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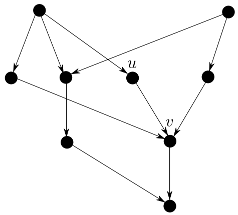
- Define graph $\overrightarrow{H}(G)$
- Max. degree of $\overrightarrow{H}(G)$: $2k-3$
- Longest directed chain of $\overrightarrow{H}(G)$: $k-1$
- Each component has a non-forced vertex
- $\Rightarrow c(k) \leq \sum_{i=0}^{k-2}(2k-3)^i$



$u \rightarrow v \Leftrightarrow N[v] = N[u] \cup \{x\}$

Corollaries

Theorem (F., Perarnau, 2011+)

There exists an integer d_0 such that for each identifiable graph G on n vertices having maximum degree $d \geq d_0$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot f(G)^2}{85d}$$

Corollary

- In general, $f(G) \geq \frac{1}{d+1}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$
- If G is d-regular, $f(G) = 1$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{85d}$.
- If G has clique number bounded by k, $f(G) \geq \frac{1}{c(k)}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d)}$.

Where are most of the d-regular graphs?

Let G be a d-regular graph.

$$\gamma^{\text{ID}}(G) \geq \frac{2n}{d+2} \qquad \text{Karpovsky et al. (1998)}$$

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1) \qquad \text{Conjecture (2009)}$$

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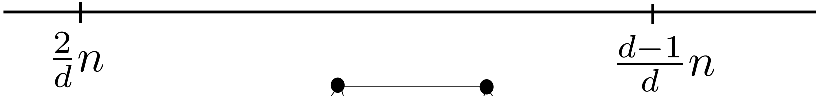

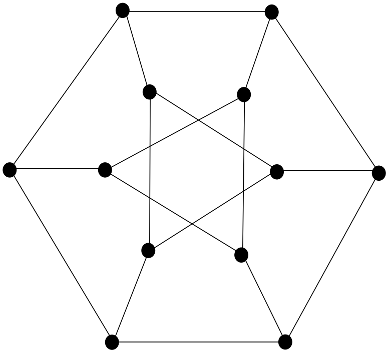

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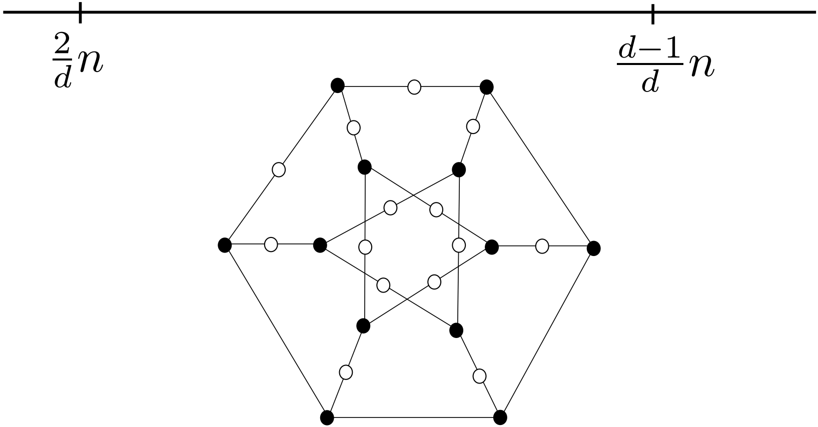

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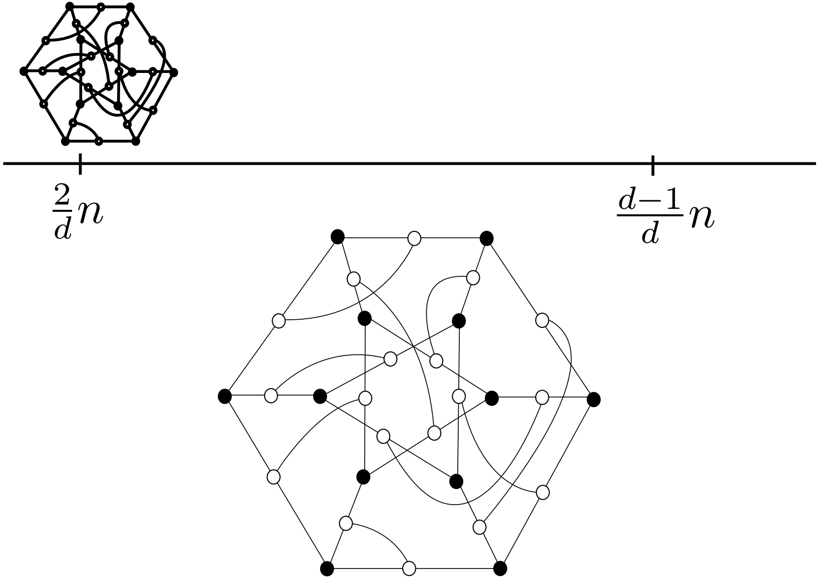

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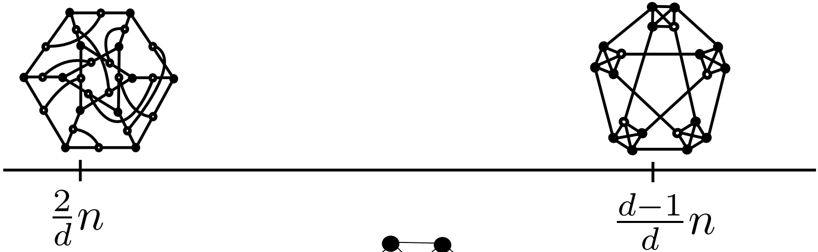

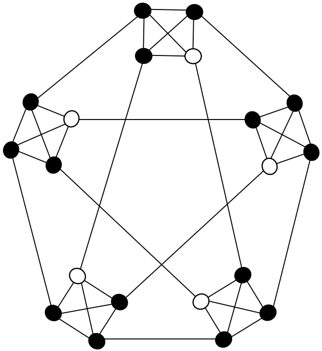

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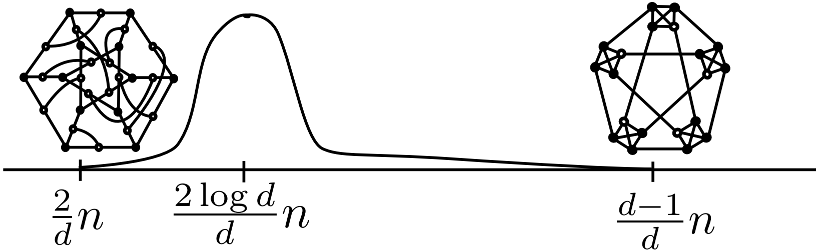

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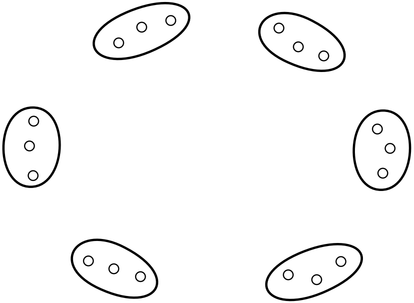
Let G be a random d-regular graph. Then a.a.s.

$$(1 + o_d(1))\frac{\log d}{d} n \leq \gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d} n$$

Probability space $\mathcal{G}^*_{n,d}$ of d-regular multigraphs on n vertices.



- Take nd vertices grouped in n buckets of size d

The pairing model (a.k.a. configuration model) - Bollobás, 1980

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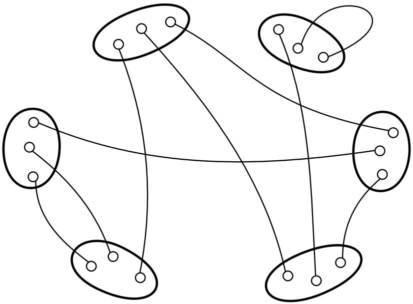

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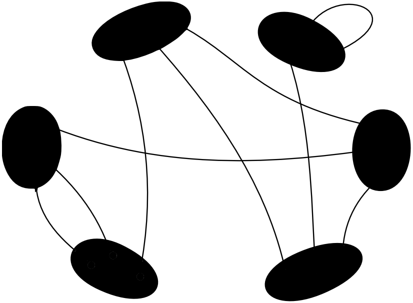

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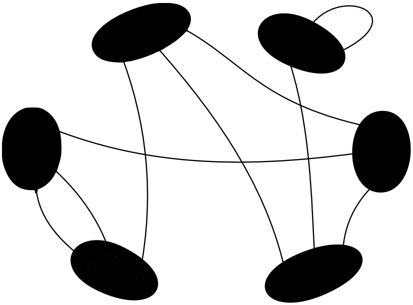

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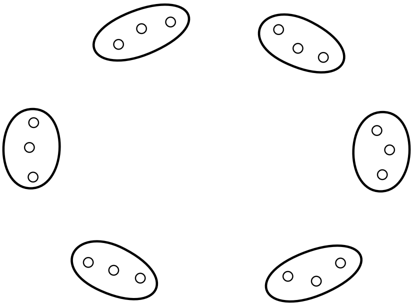

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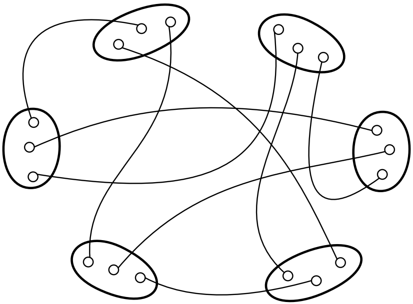

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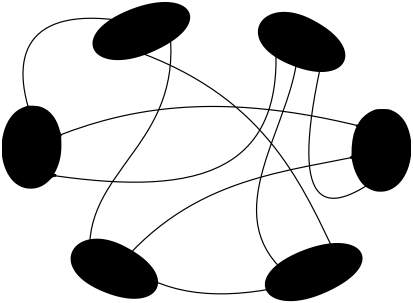

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Let $G \in \mathcal{G}^*_{n,d}$. Then $Pr(G \text{ is simple}) \longrightarrow e^{\frac{1-d^2}{4}} > 0$

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Let G be a d-regular graph with girth at least 5. Then

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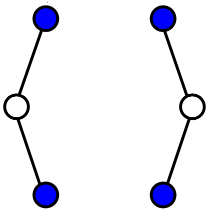

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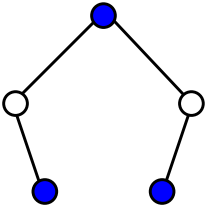

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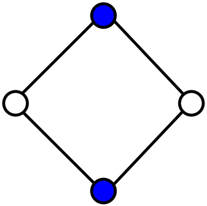

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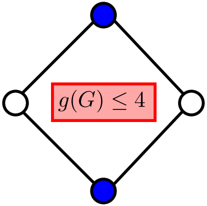

$g(G) \geq 5$ makes identifying easier.

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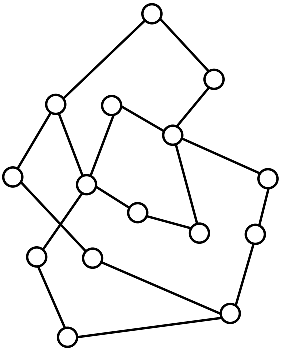

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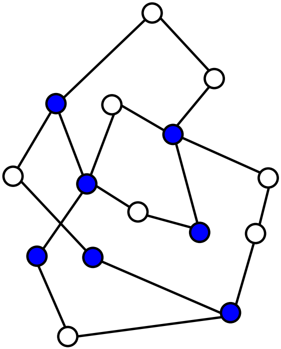

Sketch of the proof: construct 2-dominating set D

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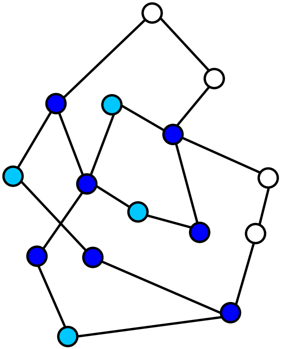

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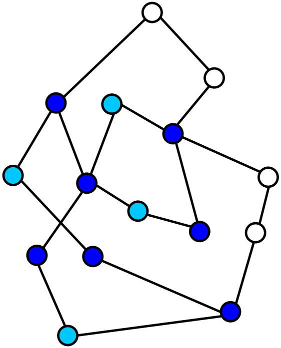

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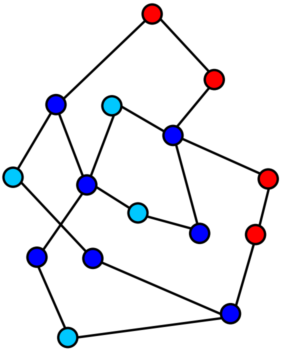

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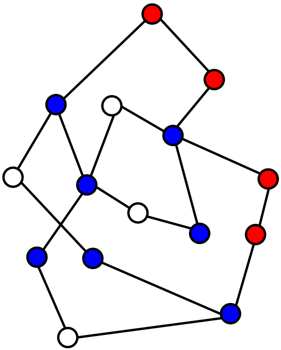
- $\mathcal{C} = S \cup \{v : X_v = 1\}$, $p = \frac{\log d}{d}$

  $$\mathbb{E}(|D|) = \mathbb{E}(|S|) + X(S) \leq \frac{2 \log d}{d} n$$



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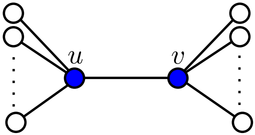
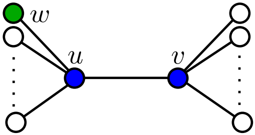


PROBLEM!

Sketch of the proof: identifying code



$SOLVED!$

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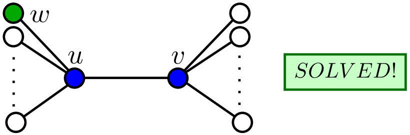

$$Y_{uv} = \begin{cases} 1 & \text{if } \\ 0 & \text{otherwise} \end{cases}$$

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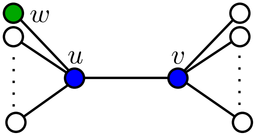

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$$\mathcal{C} = S \cup \{v : X_v = 1\} \cup \{w : w \in N(u),\ Y_{uv} = 1\},\ p = \frac{\log d}{d}$$

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Theorem (F., Perarnau, 2011+)

Let G be a random d-regular graph. Then a.a.s.

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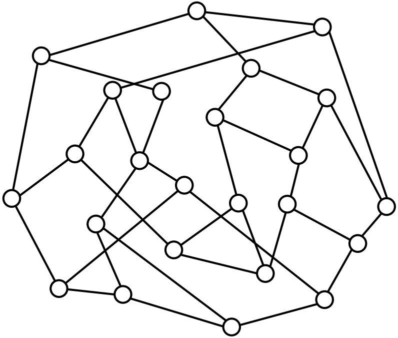
Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n, d)$



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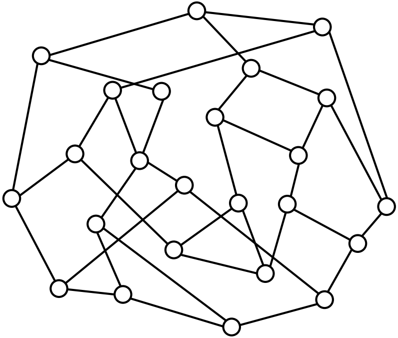
$$\Pr(G \text{ identifiable}) \xrightarrow{n} 1$$



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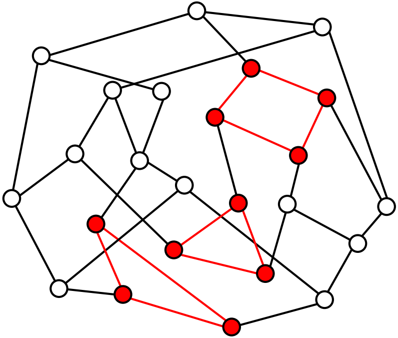
$$\gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d} n$$

Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n, d)$

$$\Pr(G \text{ identifiable}) \xrightarrow{n} 1$$

$\mathbb{E}(C_3\text{'s}) = e^{\frac{(d-1)^3}{6}}$ $\mathbb{E}(C_4\text{'s}) = e^{\frac{(d-1)^4}{8}}$



Back to random regular graphs - upper bound

Theorem (F., Perarnau, 2011+)

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$$\gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d}n$$

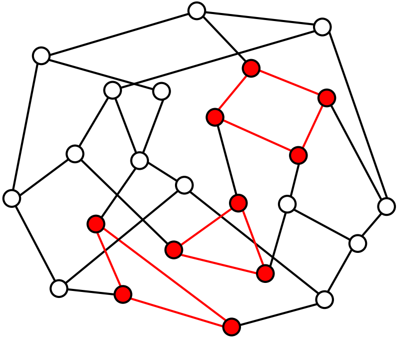
Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n,d)$

$$\Pr(G \text{ identifiable}) \xrightarrow{n} 1$$

$$\mathbb{E}(C_3\text{'s}) = e^{\frac{(d-1)^3}{6}} \qquad \mathbb{E}(C_4\text{'s}) = e^{\frac{(d-1)^4}{8}}$$

$$\Pr(\#C_3 > \log\log n) \longrightarrow 0$$

$$\Pr(\#C_4 > \log\log n) \longrightarrow 0$$



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$$\gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d}n$$

Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n,d)$

$$\Pr(G \text{ identifiable}) \xrightarrow{n} 1$$

$$\mathbb{E}(C_3\text{'s}) = e^{\frac{(d-1)^3}{6}} \qquad \mathbb{E}(C_4\text{'s}) = e^{\frac{(d-1)^4}{8}}$$

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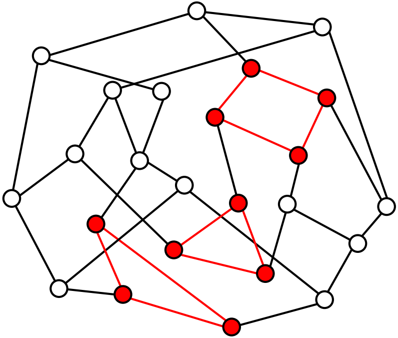

$\gamma^{\text{ID}}(G) \leq |\mathcal{C}| = (1 + o_d(1))\frac{2\log d}{d}n$ **a.a.s.**

Kiitos