10 years of

Identification problems in (hyper)graphs

selected topics

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based on joint works with:

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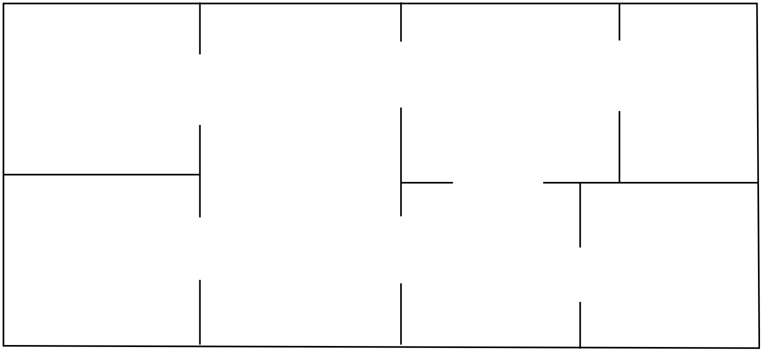
LABRI, November 2019

Identification problems

Locating a burglar



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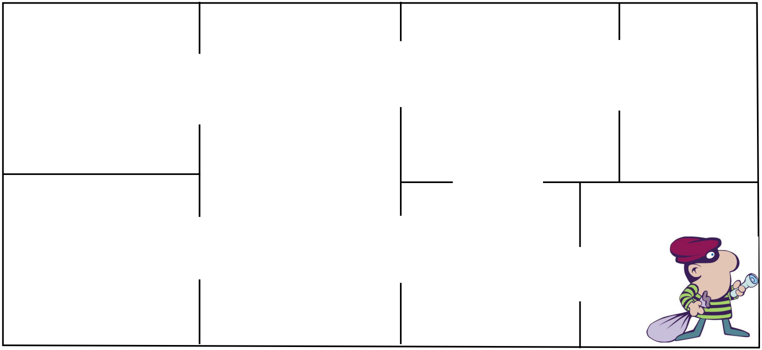

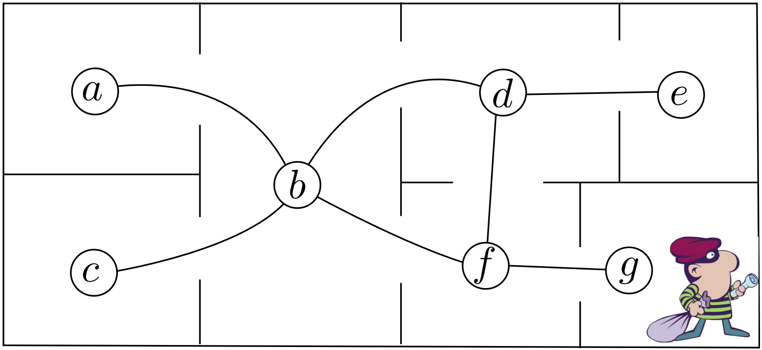
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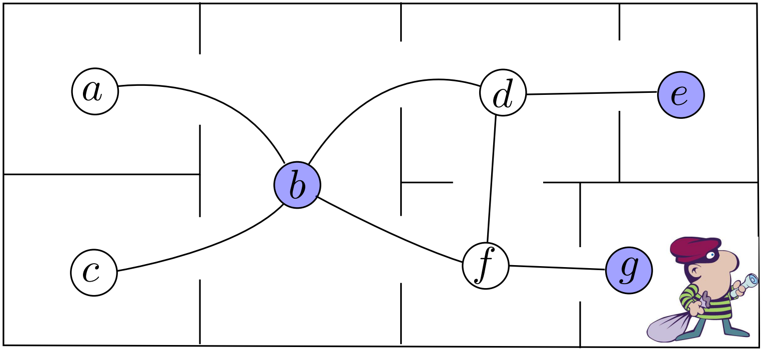
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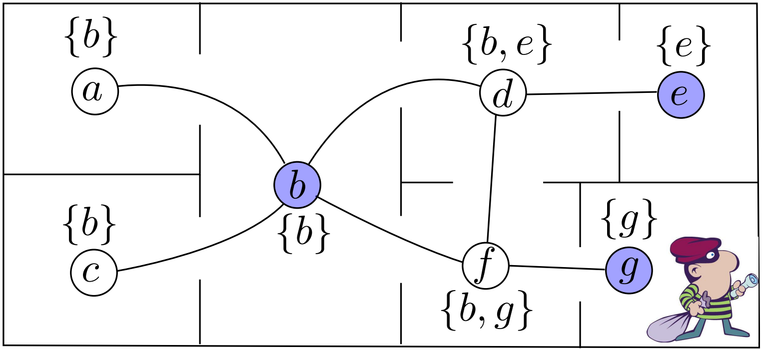
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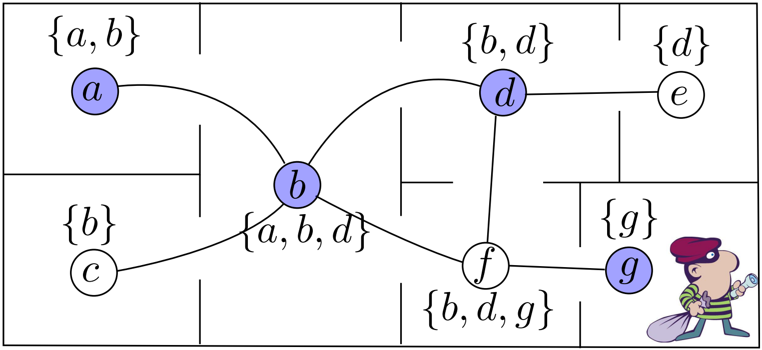
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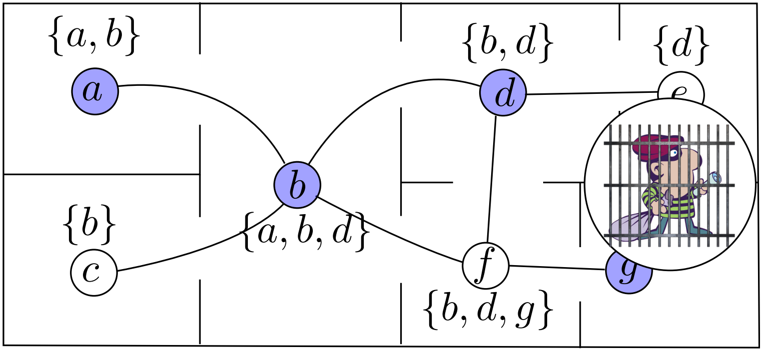
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Definition - Separating system (Rényi, 1961)

Hypergraph $(X, \mathscr{E})$. A separating system is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

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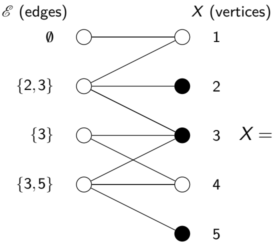

Example:

$X = \{1,2,3,4,5\}$; $\mathscr{E} = \{\{1\},\{1,2,3\},\{3,4\},\{3,4,5\}\}$

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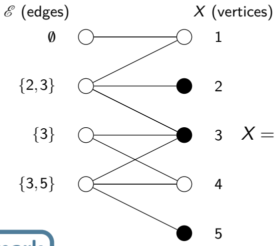

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Remark

Equivalently: for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f

General bounds

Theorem (Folklore)

For a hypergraph $(X, \mathscr{E})$, a separating system has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. $\square$

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Example with $|X|=|\mathscr{E}|+1$
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It is best possible
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Theorem (Bondy's theorem, 1972)

For a hypergraph $(X, \mathscr{E})$, a minimal separating system has size at most $|X| - 1$.

Proof: Note: if $E_1, E_2 \subseteq X$ and $E_1 - x = E_2 - x$, then $E_1 \Delta E_2 = \{x\}$.
Construct a graph H on vertex set $\mathscr{E}$ where for each $x \in X$, choose (at most) one unique pair E_i, E_j of $\mathscr{E}$ s.t. $E_i = E_j + x$, and connect E_i to E_j.
Claim: H has no cycle.
So there are at most $|X| - 1$ "forbidden" elements of X, and there is $x_0 \in X$ s.t. $X - x_0$ works. □

Some example problems

Special graph-based cases of separating sets in hypergraphs:

- *identifying codes*
- identifying open codes
- path identifying covers
- cycle identifying covers
- separating path systems
- geometric versions: e.g. seperating points using disks in Euclidean space

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A variation:

- **locating-dominating sets**
- locating-total dominating sets

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A variation:

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Distance-based identification:

- **resolving sets** (metric dimension)
- centroidal locating sets
- tracking paths problem

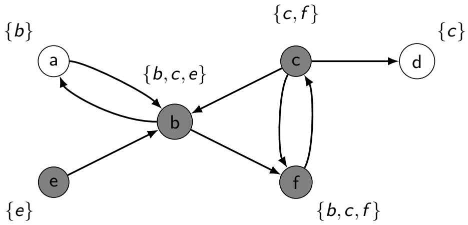
Identifying codes in digraphs

$N^-[u]$: *in-neighbourhood* of u

Definition - Identifying code of a digraph $D=(V,A)$

subset C of V such that:

- C is a **dominating set** in D: for all $u \in V$, $N^-[u] \cap C \neq \emptyset$, and
- C is a **separating code** in D: for all $u \neq v$, $N^-[u] \cap C \neq N^-[v] \cap C$

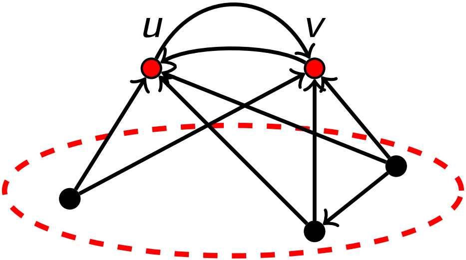


$ID(D)$: minimum size of an identifying code of D

Remark

Not all digraphs have an identifying code!

Closed in-twins = pair u, v such that $N^-[u] = N^-[v]$.



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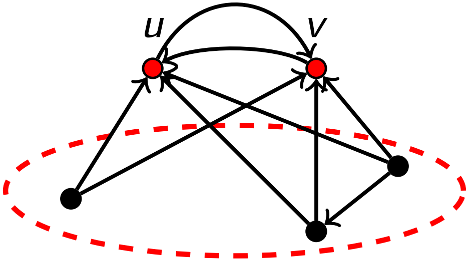

Proposition

A digraph is **identifiable** if and only if it is **closed in-twin-free** (i.e. has no closed in-twins).

Bounds

Theorem (Folklore)

G identifiable digraph on n vertices:

$$\lceil \log_2(n+1) \rceil \leq ID(D) \leq n$$

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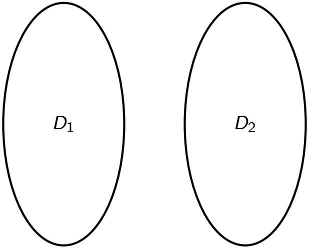
Question

Which digraphs D have $ID(D) = n$?

Which digraphs need n vertices?

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\triangleleft}(D)$: D joined to K_1 by incoming arcs only



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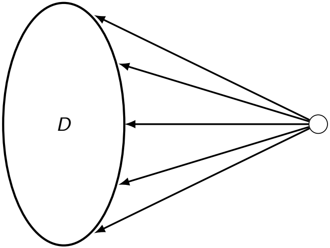

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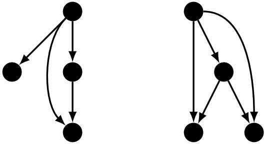

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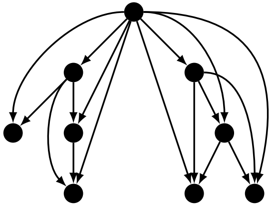

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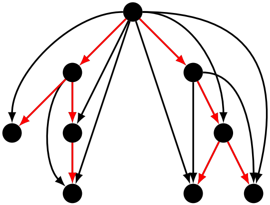
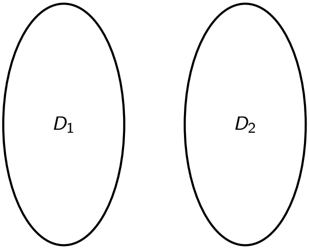


Proposition

For each digraph D of order n in $(K_1, \oplus, \overrightarrow{\triangleleft})$, $ID(D) = n$.



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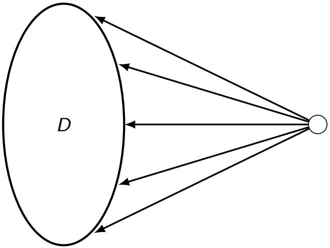
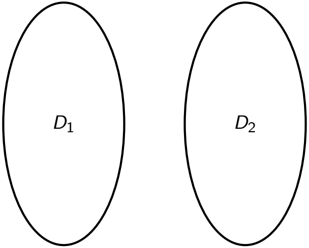


$\overrightarrow{\triangleleft}(D)$

A characterization

Theorem (F., Naserasr, Parreau, 2013)

Let D be an identifiable digraph on n vertices. $ID(G) = n$ iff $D \in (K_1, \oplus, \vec{\triangleleft})$.



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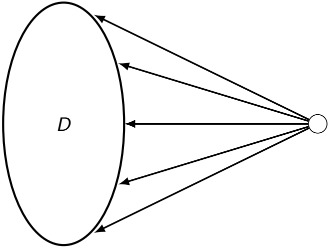

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Proof of the theorem.

Proposition

Let D be a digraph with $ID(D) = |V(D)|$, then there is a vertex x of D such that $ID(D - x) = |V(D - x)|$.

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Proposition

Let D be a digraph with $ID(D) = |V(D)|$, then there is a vertex x of D such that $ID(D - x) = |V(D - x)|$.

- By contradiction: take a minimum counterexample, D
- By the proposition, there is a vertex x such that $ID(D - x) = |V(D - x)| - 1$. By minimality of D, $D - x \in (K_1, \oplus, \overrightarrow{\triangleleft})$.
- Show that in any way of adding a vertex to $D - x$, we either stay in the family or decrease ID. □

Theorem (Bondy's theorem, 1972)

For a hypergraph $(X, \mathscr{E})$, a minimal separating system has size at most $|X| - 1$.

Remark

$B = B(X, \mathscr{E})$: bipartite graph representing $(X, \mathscr{E})$. If B has a matching from $\mathscr{E}$ to X, then B is the neighbourhood graph of a digraph D.
$\Rightarrow$ Any separating system of $(X, \mathscr{E})$ is a separating code of D.

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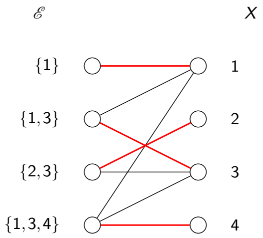

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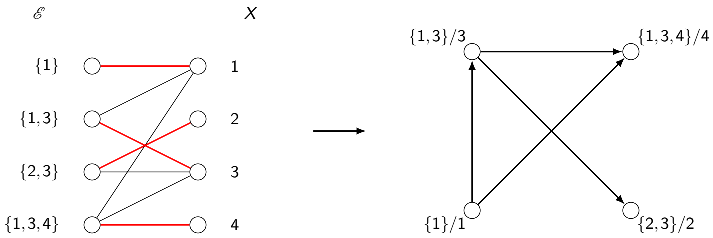

Corollary (F., Naserasr, Parreau, 2013)

In Bondy's theorem (with $|X| = |\mathscr{E}|$ and non-empty sets), if for any good choice of x we have $E_i - x = \emptyset$ for some E_i,
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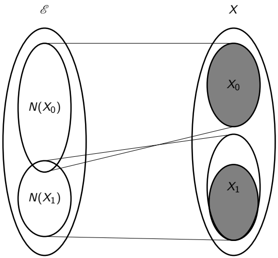
Proof:

- If B has a perfect matching: use our theorem.
- Otherwise, by Hall's theorem, there is a subset X_1 of X s.t. $|X_1| > |N(X_1)|$.



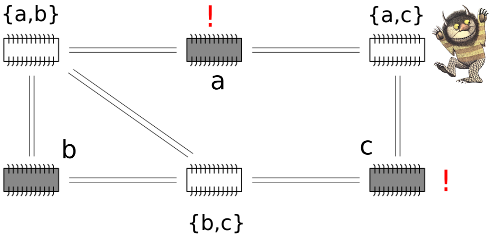
Location-domination in graphs

Definition - Locating-dominating set (Slater, 1980's)

$D \subseteq V(G)$ locating-dominating set of G:

- for every $u \in V$, $N[v] \cap D \neq \emptyset$ (domination).
- $\forall u \neq v$ of $V(G) \setminus D$, $N(u) \cap D \neq N(v) \cap D$ (location).

Motivation: fault-detection in networks.
$\rightarrow$ The set D of grey processors is a set of fault-detectors.



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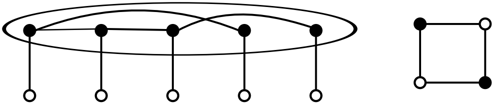
Domination number: $DOM(P_n) = \lceil \frac{n}{3} \rceil$

Identifying code number: $ID(P_n) = \lceil \frac{n+1}{2} \rceil$

Location-domination number: $LD(P_n) = \lceil \frac{2n}{5} \rceil$

Theorem (Domination bound, Ore, 1960's)

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Tight examples: 

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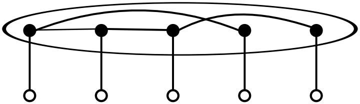
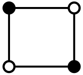
Tight examples:  

Theorem (Location-domination bound, Slater, 1980's)

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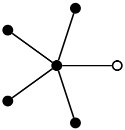
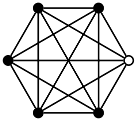
Tight examples:  

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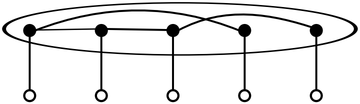
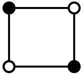
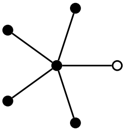
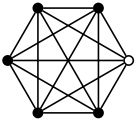
Tight examples:  

Theorem (Location-domination bound, Slater, 1980's)

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Tight examples:  

Remark: tight examples contain many twin-vertices!!

Upper bound: a conjecture

Theorem (Domination bound, Ore, 1960's)

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Theorem (Location-domination bound, Slater, 1980's)

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

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Remark:

- twins are easy to detect
- twins have a trivial behaviour w.r.t. location-domination

Theorem (Domination bound, Ore, 1960's)

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

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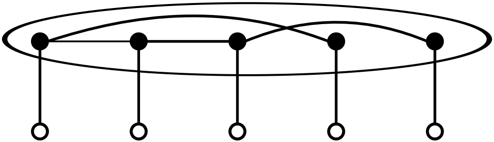
Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 1. domination-extremal graphs



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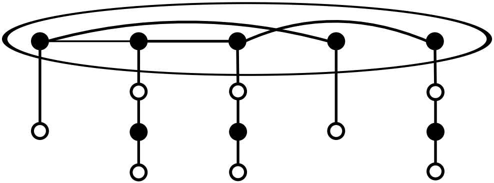
Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 2. a similar construction



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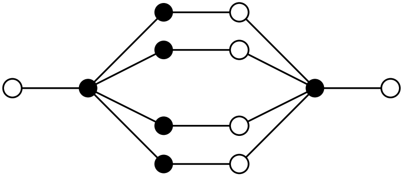
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 3. a family with domination number 2



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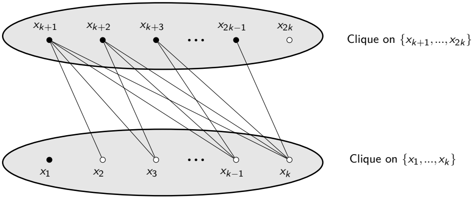
Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 4. a *dense* family with domination number 2



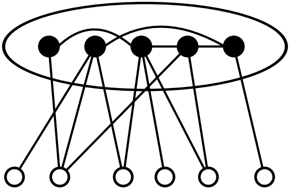
Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (Garijo, González & Márquez, 2014)

Conjecture true if G has independence number $\geq n/2$.
(in particular, if bipartite)

Proof: every vertex cover is a locating-dominating set



Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

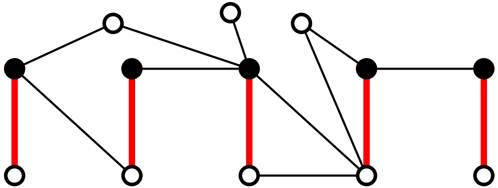
$\alpha'(G)$: matching number of G

Theorem (Garijo, González & Márquez, 2014)

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Proof:

- Consider special maximum matching M
- Select one vertex in each edge of M



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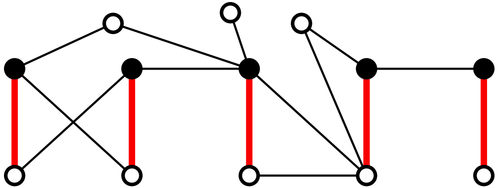


Upper bound: a conjecture - special graph classes

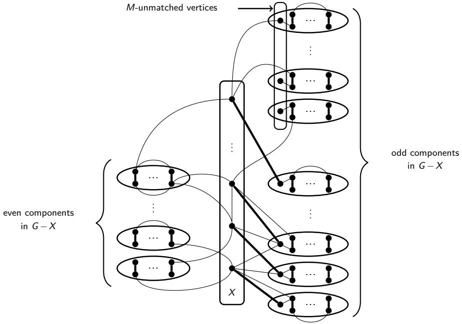
Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016)

Conjecture true if G is cubic.

Proof: Involved argument using maximum matching and Tutte-Berge theorem.



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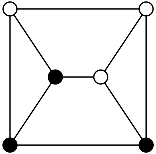
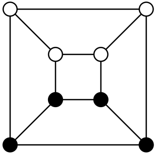
Theorem (F., Henning, 2016)

Conjecture true if G is cubic.

Bound is tight:





Question

Do we have $LD(G) = \frac{n}{2}$ for other cubic graphs?

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016)

Conjecture true if G is cubic.

Question

Are there twin-free (cubic) graphs with $LD(G) > \alpha'(G)$?

(if not, conjecture is true)

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016)

Conjecture true if G is split graph or complement of bipartite graph.

Line graph of G: intersection graph of the edges of G.

Theorem (F., Henning, 2017)

Conjecture true if G is line graph.

Proof: By induction on the order, using edge-locating-dominating sets

Upper bound: a conjecture - general bound

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

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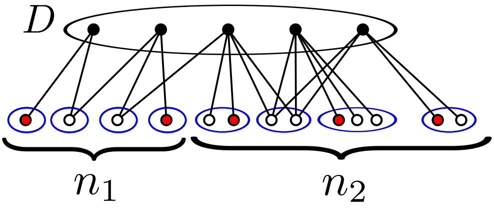
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

Proof: • There exists a dominating set D such that each vertex has a private neighbour. We have $|D| \leq n_1 + n_2$.



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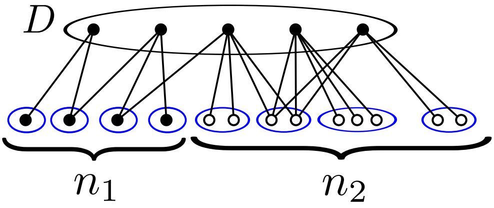
• there is a LD-set of size $|D| + n_1$;



Upper bound: a conjecture - general bound

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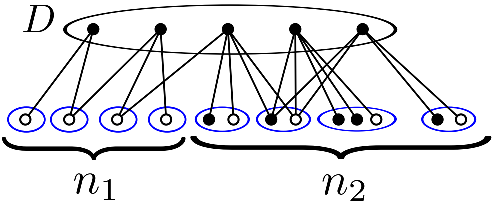
• there is a LD-set of size $|D| + n_1$;   there is a LD-set of size $n - n_1 - n_2$



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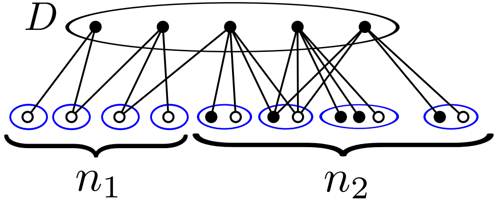
• there is a LD-set of size $|D| + n_1$;   there is a LD-set of size $n - n_1 - n_2$

• $\min\{|D| + n_1, n - n_1 - n_2\} \leq \frac{2}{3}n$



Lower bounds

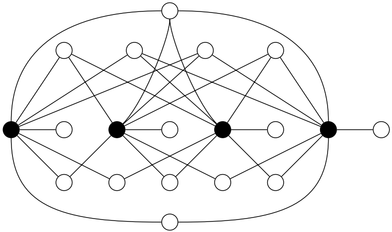
Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \rightarrow LD(G) = \Omega(\log n)$.

Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \to LD(G) = \Omega(\log n)$.

Tight example ($k = 4$):



Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \rightarrow LD(G) = \Omega(\log n)$.

Theorem (Slater, 1980's)

G tree of order n, $LD(G) = k$. Then $n \leq 3k - 1 \rightarrow LD(G) \geq \frac{n+1}{3}$.

Theorem (Rall & Slater, 1980's)

G planar graph, order n, $LD(G) = k$. Then $n \leq 7k - 10 \rightarrow LD(G) \geq \frac{n+10}{7}$.

Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \rightarrow LD(G) = \Omega(\log n)$.

Theorem (Slater, 1980's)

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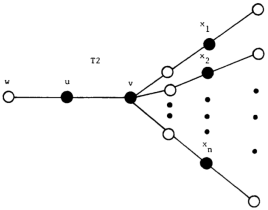


FIG. 2. Tree T2

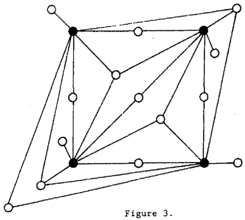


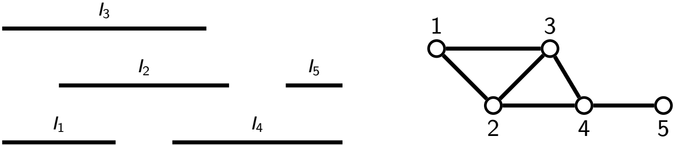
Figure 3.

Tight examples:

Interval graphs

Definition - Interval graph

Intersection graph of intervals of the real line.



Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

G interval graph of order n, $LD(G)=k$.

Then $n \leq \frac{k(k+3)}{2}$, i.e. $LD(G)=\Omega(\sqrt{n})$.

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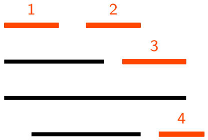

- Locating-dominating D of size k.
- Define zones using the right points of intervals in D.

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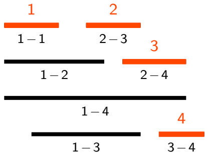

- Locating-dominating D of size k.
- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

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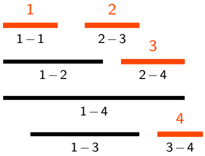


- Locating-dominating D of size k.
- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

$\rightarrow n \leq \sum_{i=1}^{k}(k-i) + k = \frac{k(k+3)}{2}$.

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

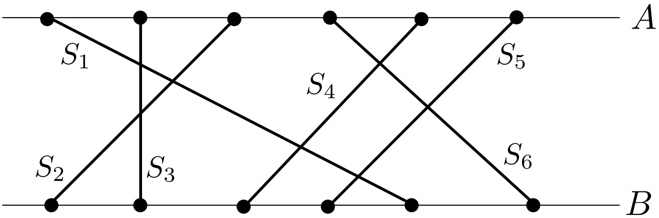
G interval graph of order n, $LD(G) = k$.

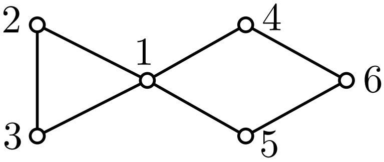
Then $n \leq \frac{k(k+3)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.

Tight:

Definition - Permutation graph

Given two parallel lines A and B:
intersection graph of segments joining A and B.





Lower bound for permutation graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

G permutation graph of order n, $LD(G) = k$.
Then $n \leq k^2 + k - 2$, i.e. $LD(G) = \Omega(\sqrt{n})$.

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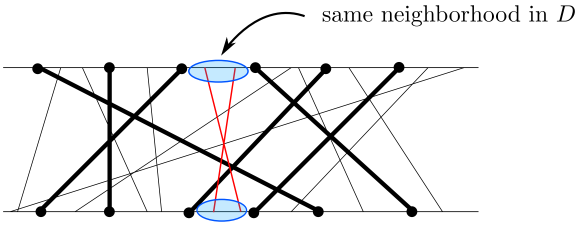

- Locating-sominating set D of size k: $k+1$ "top zones" and $k+1$ "bottom zones"
- Only one segment in $V \setminus D$ for one pair of zones

  $\rightarrow n \leq (k+1)^2 + k$
- Careful counting for the precise bound

Lower bound for permutation graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

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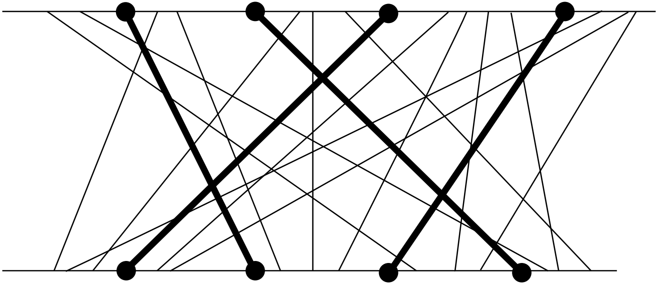
Tight:



Bounds for subclasses of interval/permutation

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

Let G be a graph on n vertices, $LD(G) = k$.

- If G is *unit* interval, then $n \leq 3k - 1$.
- If G is *bipartite* permutation, then $n \leq 3k + 2$.
- If G is a cograph, then $n \leq 3k$.

Vapnik-Červonenkis dimension

Measure of intersection complexity of sets in a hypergraph

In graphs: $X \subseteq V(G)$ is shattered:
for every subset $S \subseteq X$, there is a vertex v with $N[v] \cap X = S$

V-C dimension of G: maximum size of a shattered set in G

Typically bounded for geometric intersection graphs

Vapnik-Červonenkis dimension

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V-C dimension of G: maximum size of a shattered set in G

Typically bounded for geometric intersection graphs

Theorem (Bousquet, Lagoutte, Li, Parreau, Thomassé, 2015)

G graph of order n, $LD(G) = k$, V-C dimension $\leq d$. Then $n = O(k^d)$.

$\rightarrow$ interval graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

Vapnik-Červonenkis dimension

Measure of intersection complexity of sets in a hypergraph

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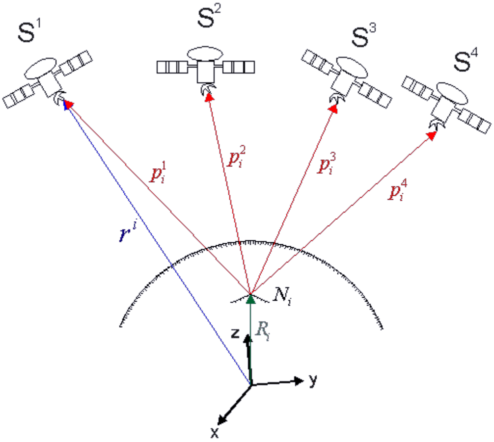
$\rightarrow$ interval graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

But better bounds exist:

- planar: $n \leq 7k - 10$ (Slater & Rall, 1984)
- line: $n \leq \frac{8}{9}k^2$ (F., Gravier, Naserasr, Parreau, Valicov, 2013)
- permutation: $n = O(k^2)$ (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

Metric dimension

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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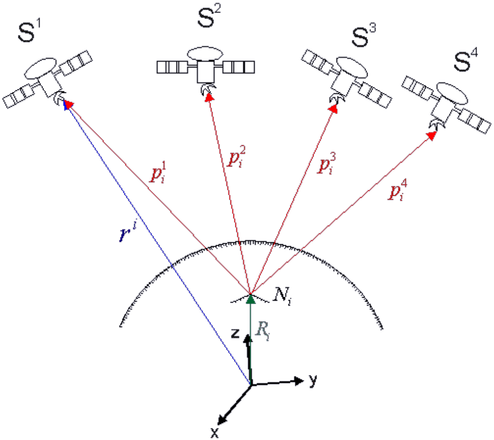

Question

Does the "GPS" approach also work in undirected unweighted graphs?

Metric dimension

Now, $w \in V(G)$ distinguishes $\{u,v\}$ if $dist(w,u) \neq dist(w,v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)

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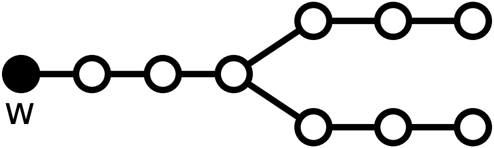

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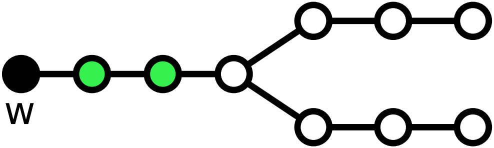

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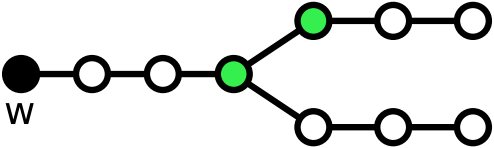

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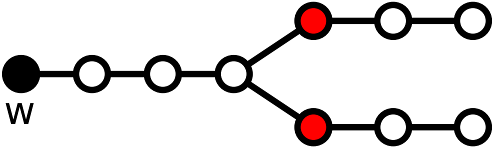

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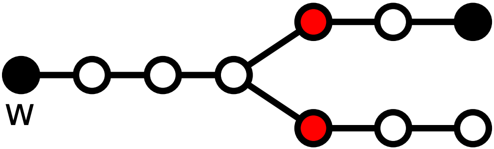

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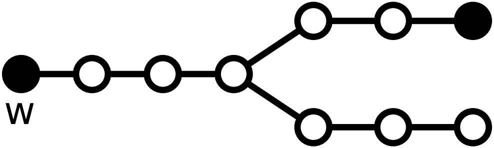

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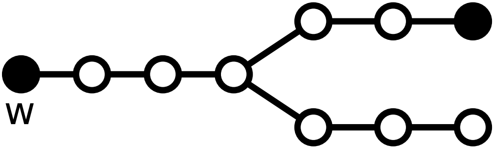
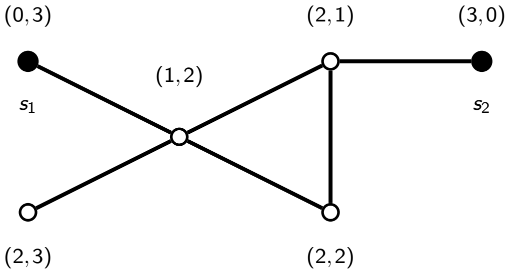


$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Example



$S = \{s_1, s_2\}$

$MD(G) = 2$

Remarks

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- Any locating-dominating set is a resolving set, hence $MD(G) \leq LD(G)$.
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Proposition

$$MD(G) = 1 \Leftrightarrow G \text{ is a path}$$

Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

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G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

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G permutation graph or interval graph of order n, $MD(G) = k$, diameter D.
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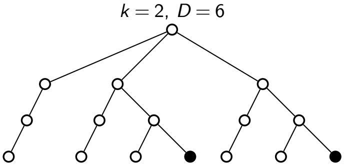
$\rightarrow$ Proofs are similar as for locating-dominating sets.

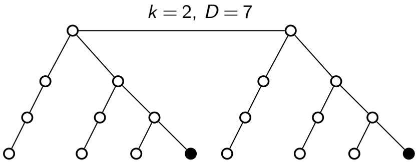
Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018)

T a tree with diameter D and $MD(T) = k$, then

$$n \leq \begin{cases} \frac{1}{8}(kD+4)(D+2) & \text{if } D \text{ even,} \\ \frac{1}{8}(kD-k+8)(D+1) & \text{if } D \text{ odd.} \end{cases} = \Theta(kD^2)$$

Bounds are tight.





Planar graphs

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G outerplanar with diameter D and $MD(G) = k$, then $n = O(kD^2)$. Tight.

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Using the concept of distance-VC-dimension:

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018)

G planar with diameter D and $MD(G) = k$, then $n = O(k^4 D^4)$.

Tight? Example with $k = 3$ and $n = \Theta(D^3)$.

THANKS FOR YOUR ATTENTION