Location-domination and metric dimension in interval and permutation graphs

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joint work with:

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April 2015


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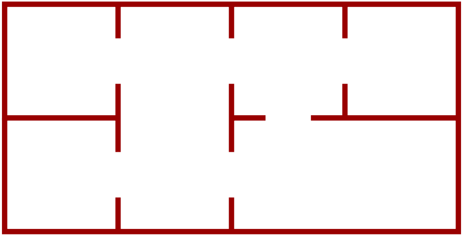
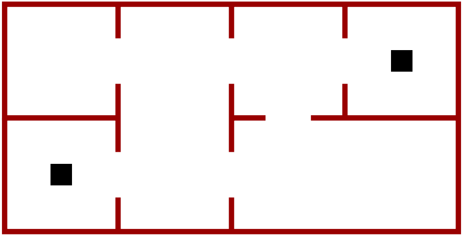
Location-domination

Fire detection in a building





- Detector can detect fire in its room and its neighborhood (through a door).

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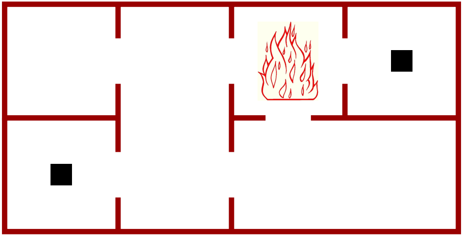

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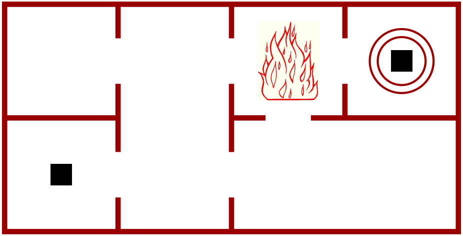

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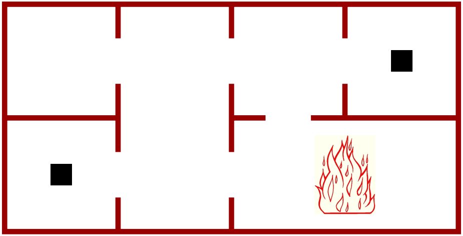

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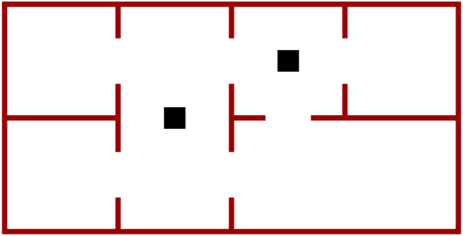

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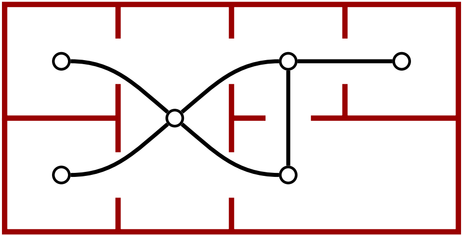
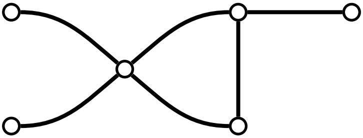


- Graph $G = (V, E)$. Vertices: rooms.
  Edges: between any two rooms connected by a door



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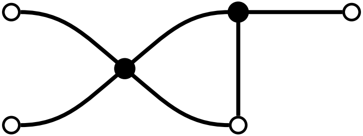

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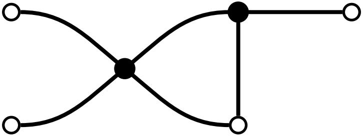
- Set of detectors = dominating set $D \subseteq V$: $\forall u \in V, N[u] \cap D \neq \emptyset$



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- Domination number $\gamma(G)$: smallest size of a dominating set of G

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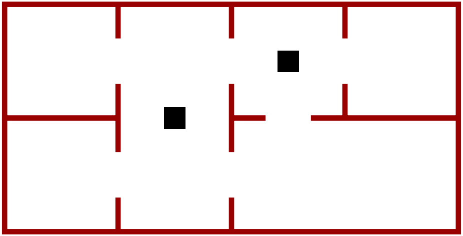

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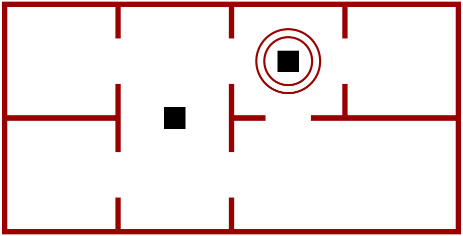

Where is the fire ?

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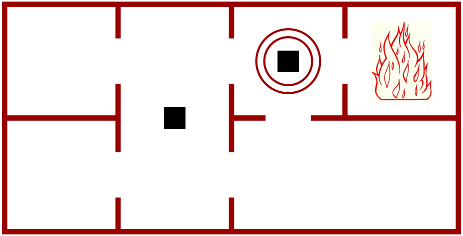

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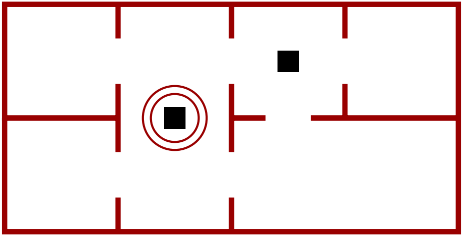

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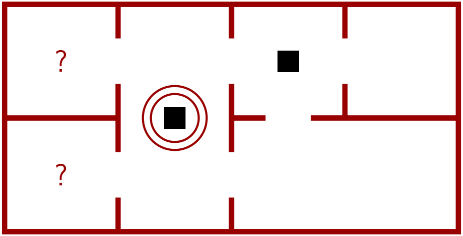
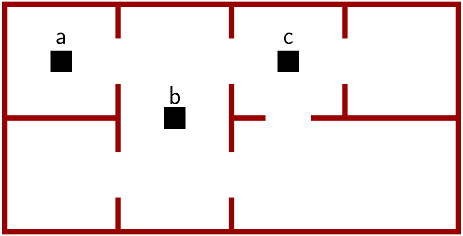


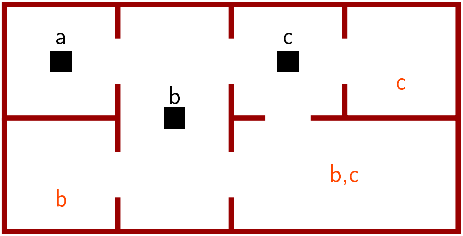
Where is the fire ?

To locate the fire, we need more detectors.

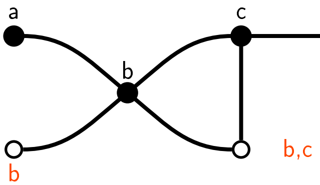
Locating the fire



Locating the fire

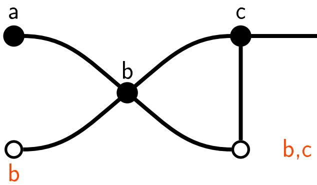


In each room with no detector, set of dominating detectors is distinct.



Peter Slater, 1980's. Locating-dominating set D:
subset of vertices of $G = (V, E)$ which is:

- dominating : $\forall u \in V, N[u] \cap D \neq \emptyset$,
- locating : $\forall u, v \in V \setminus D, N[u] \cap D \neq N[v] \cap D$.



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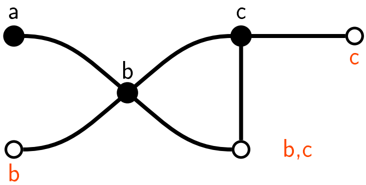
$\gamma_L(G)$: location-domination number of G,
minimum size of a locating-dominating set of G.



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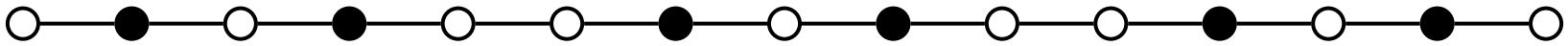
Remark: $\gamma(G) \leq \gamma_L(G)$

Examples: paths

Domination number: $\gamma(P_n) = \lceil \frac{n}{3} \rceil$

Location-domination number: $\gamma_L(P_n) = \lceil \frac{2n}{5} \rceil$



Bounds

Theorem (Slater, 1980's)

G graph of order n, $\gamma_L(G) = k$. Then $n \leq 2^k + k - 1$, i.e. $\gamma_L(G) = \Omega(\log n)$.

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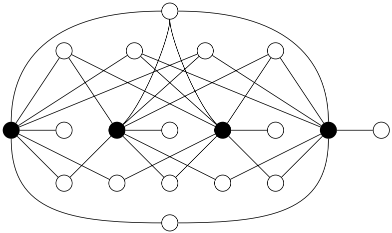
Tight example ($k = 4$):



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Theorem (Slater, 1980's)

G tree of order n, $\gamma_L(G)=k$. Then $n \leq 3k-1$, i.e. $\gamma_L(G) \geq \frac{n+1}{3}$.

Theorem (Rall & Slater, 1980's)

G planar graph, order n, $\gamma_L(G)=k$. Then $n \leq 7k-10$, i.e. $\gamma_L(G) \geq \frac{n+10}{7}$.

Tight examples:

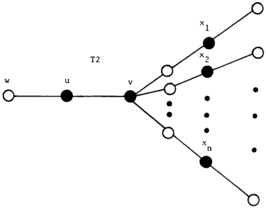


FIG. 2. Tree T2

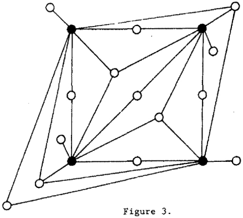
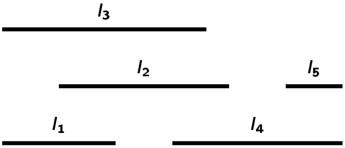
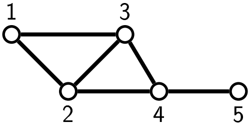


Figure 3.

Definition - Interval graph

Intersection graph of intervals of the real line.





Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

G interval graph of order n, $\gamma_L(G) = k$.

Then $n \leq \frac{k(k+3)}{2}$, i.e. $\gamma_L(G) = \Omega(\sqrt{n})$.

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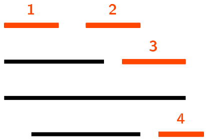

- Locating-dominating D of size k.
- Define zones using the right points of intervals in D.

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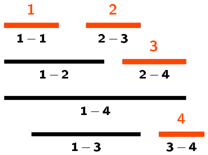

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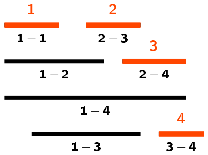

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- Define zones using the right points of intervals in D.
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$\rightarrow n \leq \sum_{i=1}^{k}(k-i) + k = \frac{k(k+3)}{2}$.

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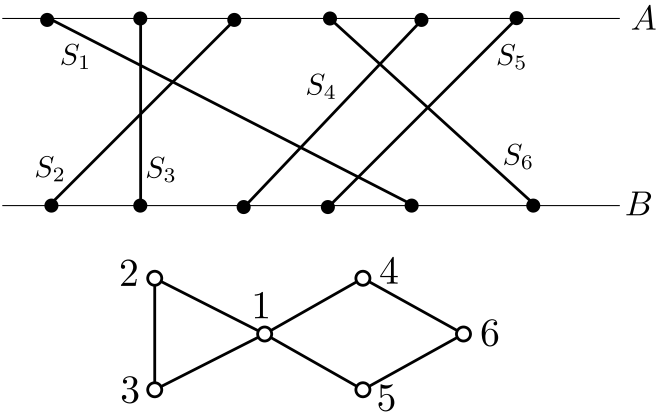
Tight:

Definition - Permutation graph

Given two parallel lines A and B:
intersection graph of segments joining A and B.



Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

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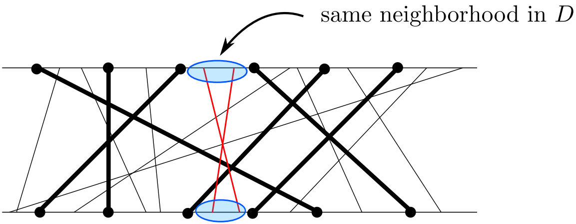

- Locating-sominating set D of size k: $k+1$ "top zones" and $k+1$ "bottom zones"
- Only one segment in $V \setminus D$ for one pair of zones

  $\rightarrow n \leq (k+1)^2 + k$
- Careful counting for the precise bound

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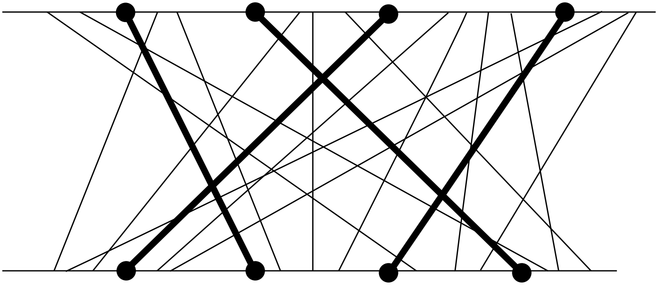
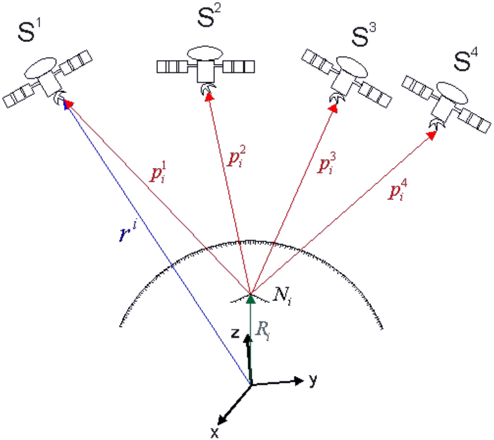


Metric dimension

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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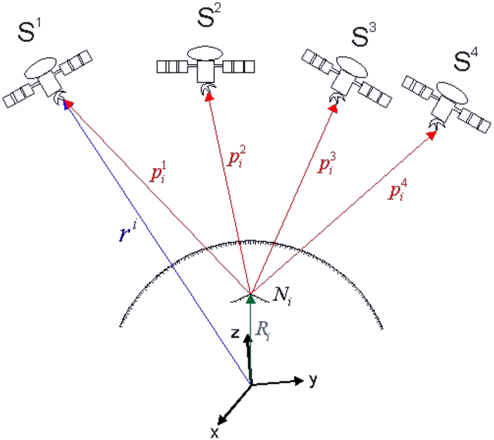

Question

Does the "GPS" approach also work in undirected unweighted graphs?

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)

$R \subseteq V(G)$ resolving set of G:

$\forall u \neq v$ in $V(G)$, there exists $w \in R$ that distinguishes $\{u, v\}$.

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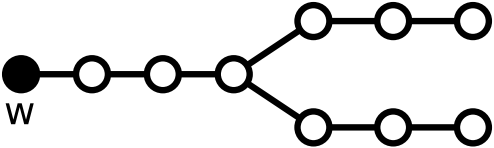

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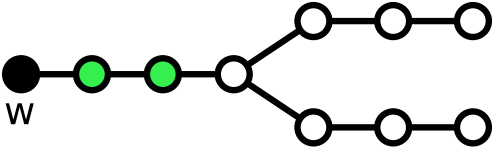

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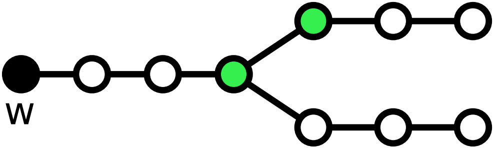

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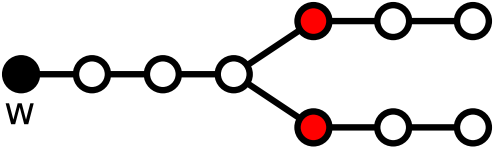

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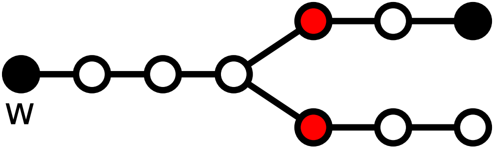

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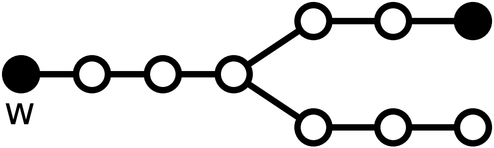

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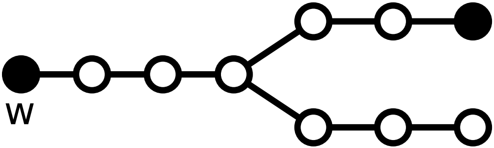

$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Remark

- Any locating-dominating set is a resolving set, hence $MD(G) \leq \gamma_L(G)$.
- A locating-dominating set can be seen as a "distance-1-resolving set".

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Proposition

$$MD(G) = 1 \Leftrightarrow G \text{ is a path}$$

Example of path: no bound $n \leq f(MD(G))$ possible.

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Theorem (Khuller, Raghavachari & Rosenfeld, 2002)

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter: maximum distance between two vertices)

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Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

G interval graph or permutation graph of order n, $MD(G) = k$, diameter D. Then $n = O(Dk^2)$ i.e. $k = \Omega\left(\sqrt{\frac{n}{D}}\right)$.

→ Proofs are similar as for locating-dominating sets.

→ Bounds are tight (up to constant factors).

Algorithmic complexity

LOCATING-DOMINATING SET

INPUT: Graph G, integer k.
QUESTION: Is there a locating-dominating set of G of size k?

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Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

LOCATING-DOMINATING SET is NP-complete for graphs that are both interval and permutation.

Reduction from 3-DIMENSIONAL MATCHING.

Main idea: an interval can separate pairs of intervals **far away** from each other (without affecting what lies in between)

METRIC DIMENSION

INPUT: Graph G, integer k.
QUESTION: Is there a resolving set of G of size k?

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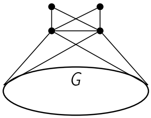
QUESTION: Is there a resolving set of G of size k?

Reduction from LOCATING-DOMINATING SET to METRIC DIMENSION:



$MD(G') = \gamma_L(G) + 2$

Corollary (F., Mertzios, Naserasr, Parreau, Valicov)

METRIC DIMENSION is NP-complete for graphs that are both interval and permutation (and have diameter 2).

Note: METRIC DIMENSION W[2]-hard even for subcubic bipartite graphs
$\longrightarrow$ probably no $f(k)poly(n)$-time algorithm

Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

METRIC DIMENSION can be solved in time $2^{O(k^4)}n$ on interval graphs.

Ideas:

- use dynamic programming on a path-decomposition of G^4.
- each bag has size $O(k^2)$.
- it suffices to separate vertices at distance 2
- "transmission" lemma for separation constraints

- Investigate bounds for other "geometric" graphs, for MD and γ_L
- Complexity of LOCATING-DOMINATING SET, METRIC DIMENSION on unit interval graphs
- Complexity of METRIC DIMENSION for bounded treewidth
- Parameterized complexity of METRIC DIMENSION: planar graphs, chordal graphs, permutation graphs...

Open problems

- Investigate bounds for other "geometric" graphs, for MD and γ_L
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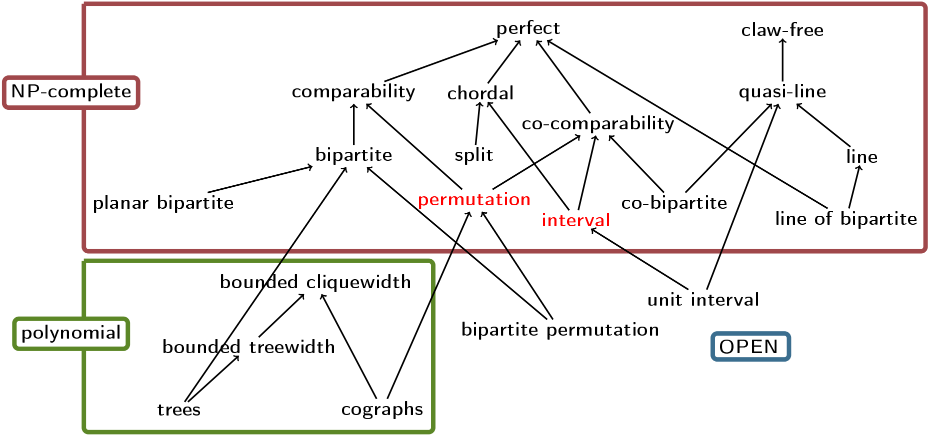
THANKS FOR YOUR ATTENTION

Complexity of LOCATING-DOMINATING SET



Complexity of METRIC DIMENSION