# Edge-identifying codes
# (identifying codes in line graphs)

**Florent Foucaud**[1]

joint work with **S. Gravier**[2], **R. Naserasr**[1], **A. Parreau**[2], **P. Valicov**[1]

1: LaBRI, Bordeaux (France)
2: Institut Fourier, Grenoble (France)

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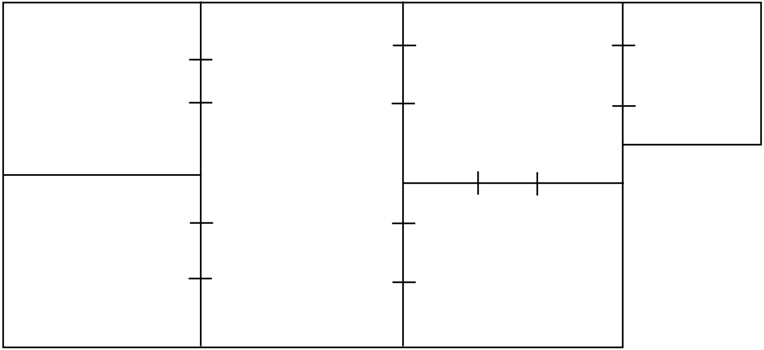
# Locating a burglar in a museum



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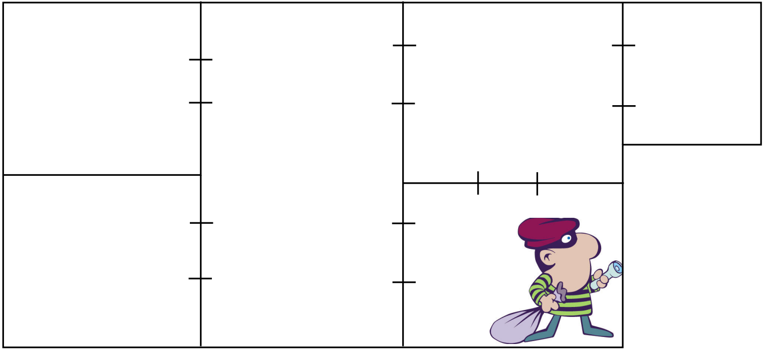

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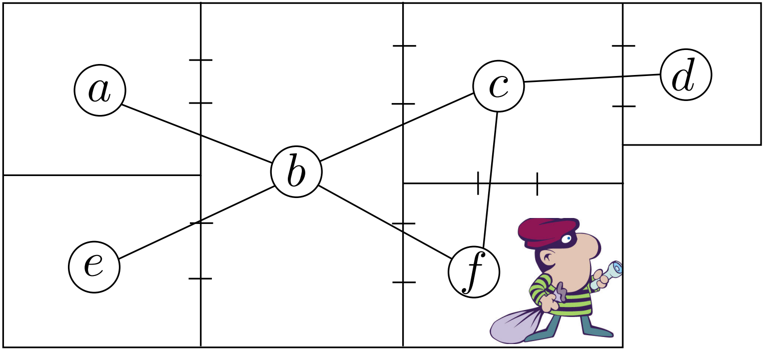

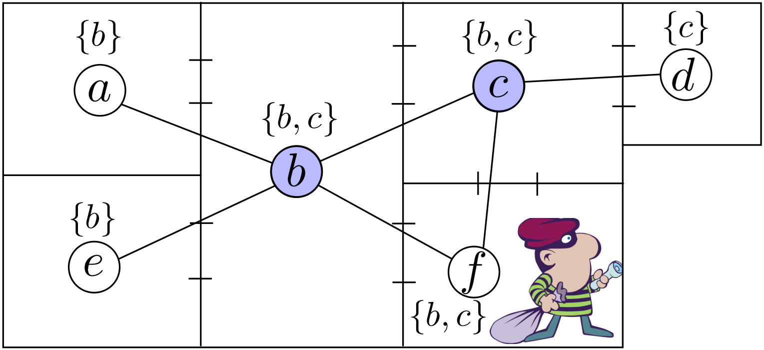
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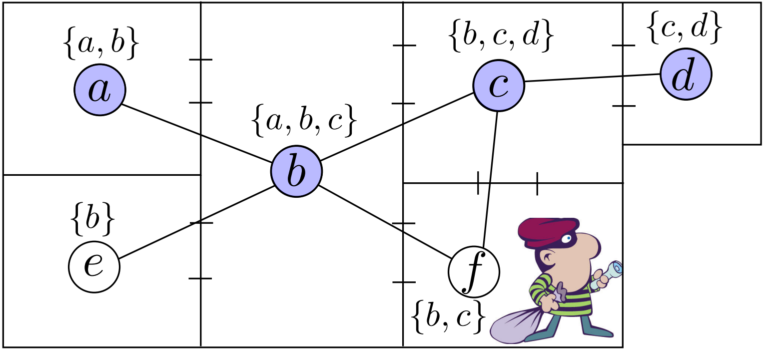
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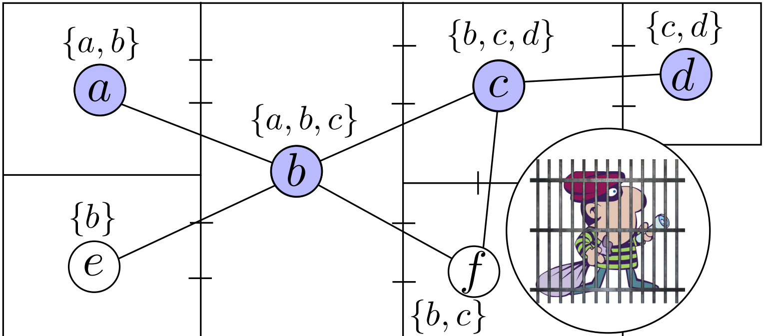

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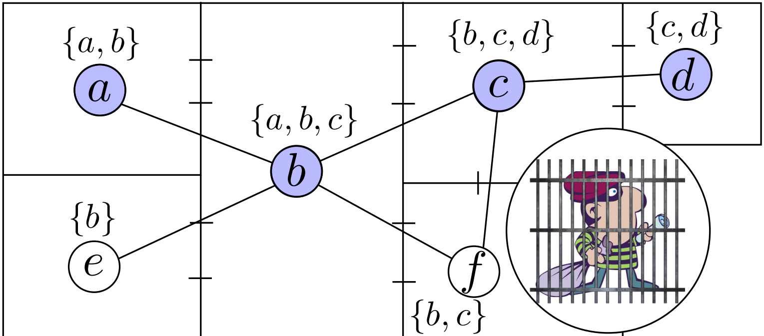


How many detectors do we need?

Let $N[u]$ be the set of vertices $v$ s.t. $d(u, v) \leq 1$

**Definition** - Identifying code of $G$ (Karpovsky, Chakrabarty, Levitin, 1998)

Subset $C$ of $V(G)$ such that:

- $C$ is a dominating set in $G$: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
- $C$ is a separating code in $G$: $\forall u \neq v$ of $V(G)$, $(N[u] \Delta N[v]) \cap C \neq \emptyset$

# Identifying codes: definition

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**Notation** - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of $G$

# Identifiable graphs

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**Remark** - Not all graphs have an identifying code!

Twins = pair $u$, $v$ such that $N[u] = N[v]$.
A graph is identifiable iff it is twin-free (i.e. it has no twins).

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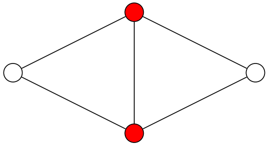

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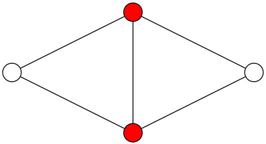

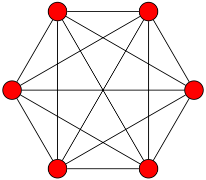

# Bounds

**Theorem** (Karpovsky, Chakrabarty, Levitin, 1998)

Let $G$ be an identifiable graph, then $\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G)$

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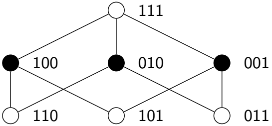

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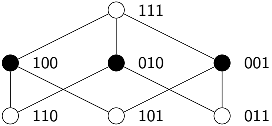


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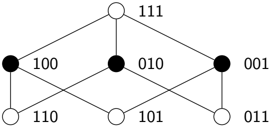


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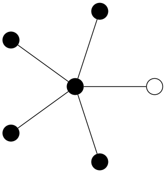

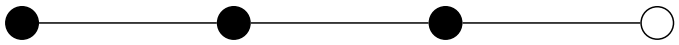

Let $N[e]$ be the set of edges $f$ s.t. $e = f$ or $e, f$ are incident to a common vertex

**Definition** - Edge-identifying code of $G$ (without isolated vertices)

Subset $C_E$ of $E(G)$ such that:

- $C_E$ is an edge-dominating set in $G$: $\forall e \in E(G)$, $N[e] \cap C_E \neq \emptyset$, and
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**Notation** - Edge-identifying code number

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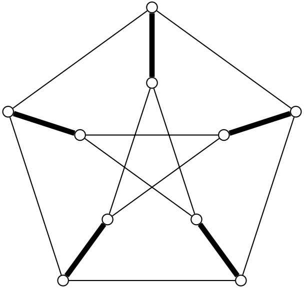
# Edge-identifying code - example



# Line graph

**Definition** - Line graph of $G$: Edge-adjacency graph of $G$

Denoted $\mathcal{L}(G)$
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$e \sim e'$ in $\mathcal{L}(G)$ iff $e$ and $e'$ are incident to a common vertex in $G$

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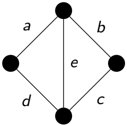

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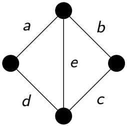


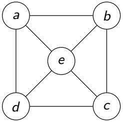

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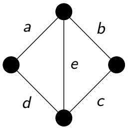
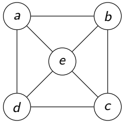


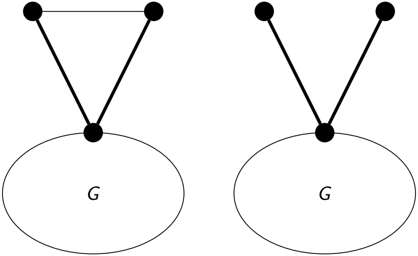

**Remark**

Edge-identifying code of $G \iff$ Identifying code of $\mathcal{L}(G)$

$$\gamma^{\text{EID}}(G) = \gamma^{\text{ID}}(\mathcal{L}(G))$$

**Remark** - Not all graphs have an edge-identifying code!

Pendant = pair of twin edges.
A graph is edge-identifiable iff it is pendant-free (and simple).



# Lower bounds

**Theorem** (F., Gravier, Naserasr, Parreau, Valicov, 2011+)

Let $G$ be an edge-identifiable graph. Then $\gamma^{\text{EID}}(G) \geq \frac{|V(G)|}{2}$.
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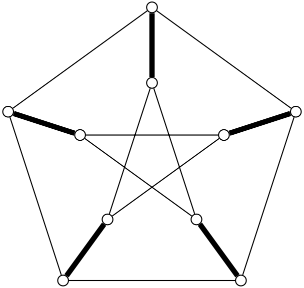

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$$\gamma^{\text{ID}}(\mathcal{L}(G)) = \gamma^{\text{EID}}(G) \geq \frac{|V(G)|}{2} \geq \frac{\sqrt{2|E(G)|}}{2} = \frac{\sqrt{2|V(\mathcal{L}(G))|}}{2}$$

# Lower bounds - proof

**Theorem** (F., Gravier, Naserasr, Parreau, Valicov, 2011+)

Let $G$ be an edge-identifiable graph. Then $\gamma^{\text{EID}}(G) \geq \frac{|V(G)|}{2}$.
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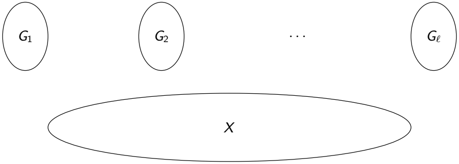
$C_E$: edge id. code of $G$.
$G_1, \ldots, G_\ell$: components of $G[C_E]$
$G_i$: $n_i$ vertices, $k_i$ edges, $n'_i$ attached vertices from $X$
$X$: vertices outside of $G[C_E]$

Claim: $\forall i, k_i \geq \frac{n_i + n' i}{2}$



# Lower bounds - improvement

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**Theorem** (F., Gravier, Naserasr, Parreau, Valicov, 2011+)

Let $G$ be an edge-identifiable graph with an edge-identifying code of size $k$,

then $|E(G)| \leq \begin{cases} \binom{\frac{4}{3}k}{2}, & \text{if } k \equiv 0 \mod 3 \\ \binom{\frac{4}{3}(k-1)+1}{2} + 1, & \text{if } k \equiv 1 \mod 3 \\ \binom{\frac{4}{3}(k-2)+2}{2} + 2, & \text{if } k \equiv 2 \mod 3 \end{cases}$

**Corollary**

$\gamma^{\text{ID}}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4}\sqrt{|V(\mathcal{L}(G))|}$. This is tight.

Extremal examples: $C_E$ = disjoint union of $P_4$'s, max. possible edges between them.

# Lower bound - question

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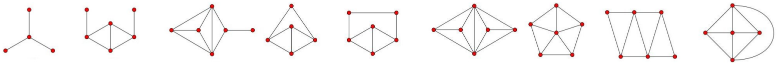
**Theorem** (Beineke, 1970)

$G$ is a line graph iff it has none of the following graphs as induced subgraph:



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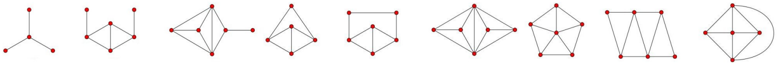

The bound does **not** hold for claw-free graphs!

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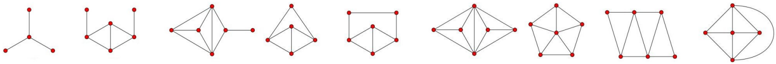

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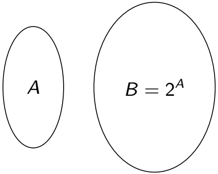
$A = \{a_1, \ldots, a_k\}$ and $B = 2^A$: cliques.
$|V(G)| = k + 2^k$
$\gamma^{\text{ID}}(G) \leq 2k = O(\log(|V(G)|))$



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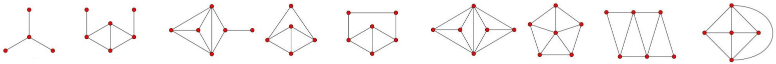

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**Question**

Does it hold for a class defined by a smaller subfamily of Beineke's list?

# An upper bound

**Definition** - $k$-degenerate graph

$G$ is $k$-degenerate if there is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that $\forall i$, $v_i$ is of degree at most $k$ in $G[v_1, \ldots, v_i]$.

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**Theorem** (F., Gravier, Naserasr, Parreau, Valicov, 2011+)

Let $G$ be an edge-identifiable graph with a minimal edge-identifying code $C_E$. Then $G[C_E]$ is 2-degenerate.

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Let $G$ be an edge-identifiable graph with a minimal edge-identifying code $C_E$. Then $G[C_E]$ is 2-degenerate.

Proof: We want to define a good ordering of $V(G)$. Let $uv \in C_E$.

- If $d(u) \leq 2$ or $d(v) \leq 2$, we are done.
- Otherwise, by minimality of $C_E$, edge $uv$ is needed to separate some pair.
- Then, there is a "local" ordering for removing either $u$ or $v$.

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Moreover, $K_4^-$ is the only graph reaching this bound.

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This is almost tight since $\gamma^{\text{EID}}(K_{2,n}) = 2n - 2 = 2|V(K_{2,n})| - 6$.

# An upper bound - corollary

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If $G$ edge-identifiable, $\gamma^{\text{EID}}(G) \leq 2|V(G)| - 3$.

**Corollary**

If $G$ is an edge-identifiable graph **with average degree** $\overline{d}(G) \geq 5$, then $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq n - \frac{n}{\Delta(\mathcal{L}(G))}$ where $n = |V(\mathcal{L}(G))|$.

**Corollary**

If $G$ edge-identifiable, $\gamma^{\text{EID}}(G) \leq 2|V(G)| - 3$.

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**Conjecture** (F., Klasing, Kosowski, Raspaud, 2009)

Let $G$ be a connected identifiable graph on $n$ vertices and of maximum degree $\Delta$. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + O(1)$.

# Complexity

**Problem EDGE-IDCODE**

INSTANCE: A graph $G$ and an integer $k$.

QUESTION: Does $G$ have an edge-identifying code of size at most $k$?

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**Theorem** (F., Gravier, Naserasr, Parreau, Valicov, 2011+)

EDGE-IDCODE is NP-complete, even for planar subcubic bipartite graphs of arbitrarily large girth.

Proof by reduction from PLANAR $(\leq 3, 3)$-SAT

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EDGE-IDCODE is NP-complete, even for planar subcubic bipartite graphs of arbitrarily large girth.

**Corollary**

IDCODE is NP-complete even when restricted to perfect 3-colorable planar line graphs of maximum degree 4.

# Thank you!

$$\frac{1}{2}|V(G)| \leq \gamma^{\text{EID}}(G) \leq 2|V(G)| - 3$$

In general: $n - 1 \geq \gamma^{\text{ID}}(G) \geq \Omega(\log n)$

In line graphs: $\gamma^{\text{ID}}(G) \geq \Omega(\sqrt{n})$.

# Advertisement: BWIC 2011

**Bordeaux Workshop on Identifying Codes (and related topics)**

21st-25th November, 2011 at the LaBRI in Bordeaux, France

`http://bwic2011.labri.fr`

Scope:

- Identifying codes
- Locating-dominating sets
- Metric dimension
- Identifying or locating colourings
- Related topics...