Edge identifying codes
(identifying codes in line graphs)

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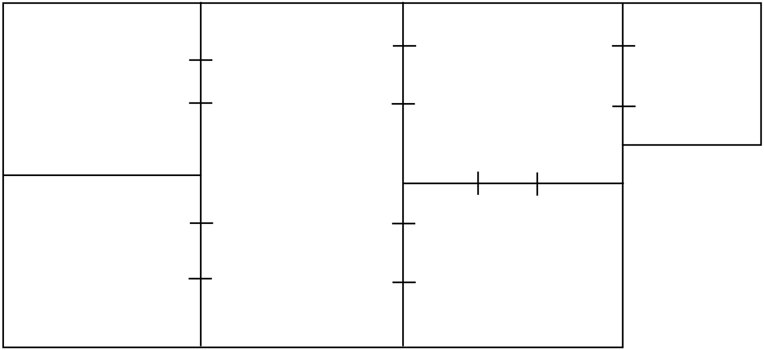
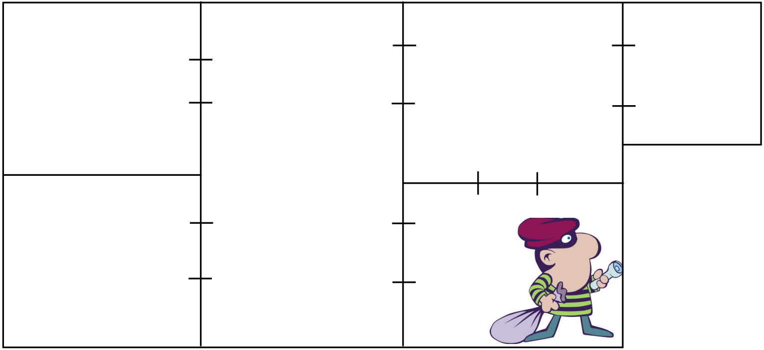
ANR IDEA



Locating a burglar in a math department



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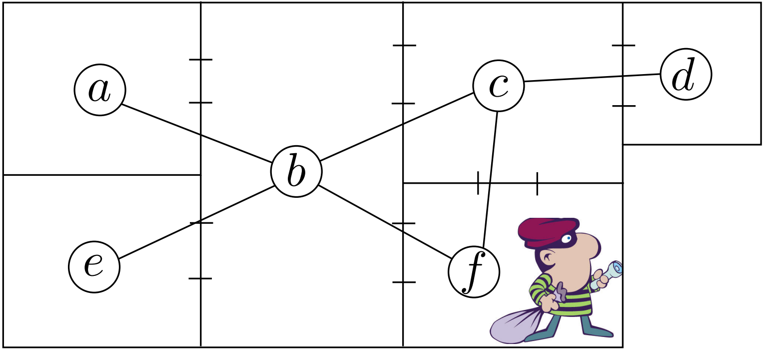

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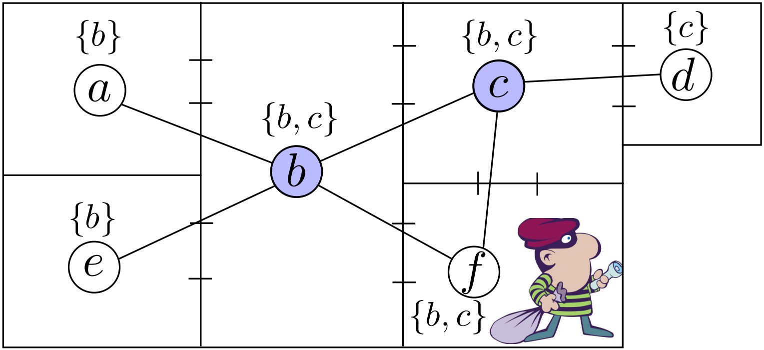

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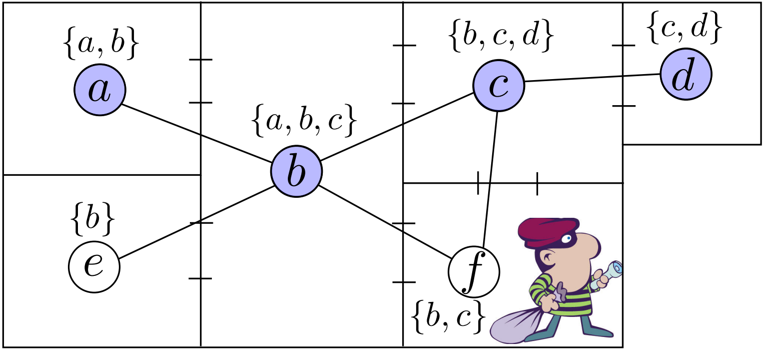

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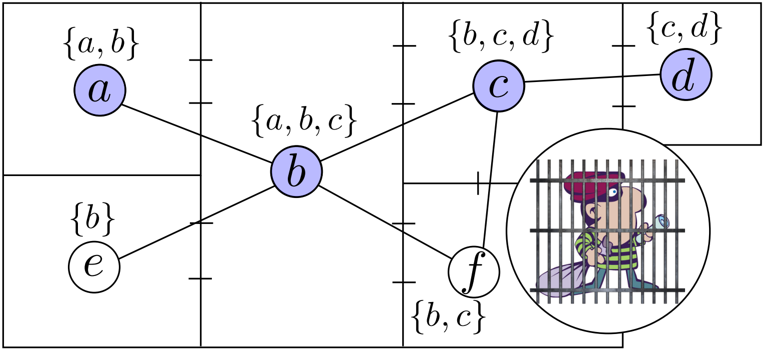

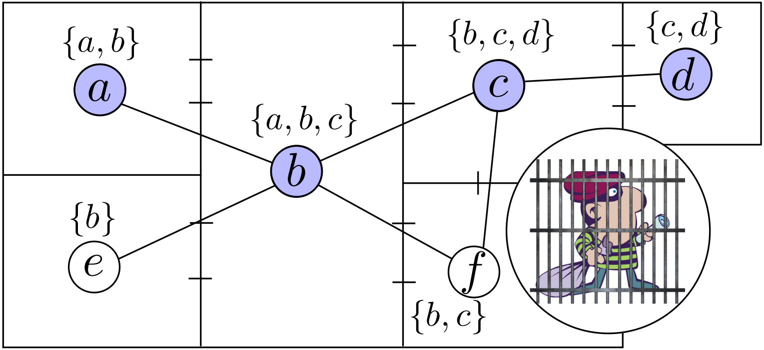

How many detectors do we need?

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a dominating set in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating code in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Remark

Not all graphs have an identifying code!

Twins $=$ pair u, v such that $N[u] = N[v]$.

A graph is identifiable iff it is twin-free (i.e. it has no twins).

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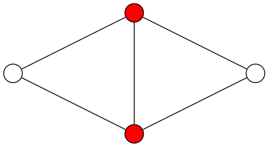

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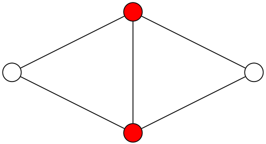

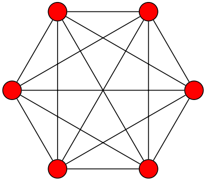

Bounds

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Both bounds are tight, and all extremal examples are known:

- lower bound: Moncel, 2006
- upper bound: F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011

Edge identifying codes, definition

Let $I[e]$ be the set of edges f s.t. $e = f$ or e, f are incident to a common vertex

Definition - Edge identifying code of G (without isolated vertices)

Subset C_E of E such that:

- C_E is an edge dominating set in G: $\forall e \in E$, $I[e] \cap C_E \neq \emptyset$, and
- C_E is an edge separating code in G: $\forall e \neq f$ of E, $I[e] \cap C_E \neq I[f] \cap C_E$

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Edge identifying code of G $\longleftrightarrow$ Identifying code of $\mathcal{L}(G)$

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Edge identifying code of $G \longleftrightarrow$ Identifying code of $\mathcal{L}(G)$

Notation - Edge identifying code number

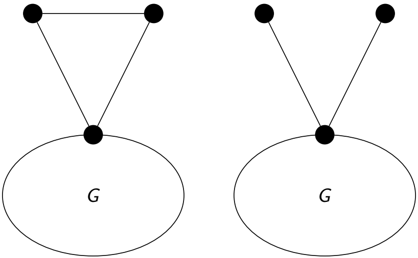
$\gamma^{\text{ID}}(\mathcal{L}(G)) = \gamma^{\text{EID}}(G)$: minimum cardinality of an edge identifying code of G

Remark

Not all graphs have an edge identifying code!

Pendant $=$ pair of twin edges.

A graph is edge identifiable iff it is pendant-free (and simple).



Theorem (F., Gravier, Naserasr, Parreau, Valicov)

Let G be an edge identifiable graph with an edge identifying code C_E **inducing a connected subgraph**, then $|E(G)| \leq \binom{|C_E|+2}{2} - 4$

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Theorem (F., Gravier, Naserasr, Parreau, Valicov)

Let G be an edge identifiable graph with an edge identifying code of size k,

then $$|E(G)| \leq \begin{cases} \binom{\frac{4}{3}k}{2}, & \text{if } k \equiv 0 \mod 3 \\ \binom{\frac{4}{3}(k-1)+1}{2} + 1, & \text{if } k \equiv 1 \mod 3 \\ \binom{\frac{4}{3}(k-2)+2}{2} + 2, & \text{if } k \equiv 2 \mod 3 \end{cases}$$

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Corollary

$\gamma^{\text{ID}}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4}\sqrt{|V(\mathcal{L}(G))|}$. This bound is tight.

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Let $G' = G[C_E]$. Each edge $uv \in G$ is determined by two sets:

- set of edges of G' incident to u
- set of edges of G' incident to v

At most $|V(G')| + \binom{|V(G')|}{2} = \binom{|V(G')|+1}{2}$ such sets.

- G' not a tree $\Rightarrow |V(G')| \leq |C_E|$
- G' tree: we show that at least 4 of these sets cannot be used.

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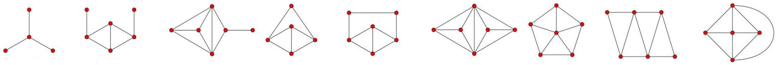
Theorem (Beineke, 1970)

G is a line graph if and only if it does not contain one of the following graphs as an induced subgraph.



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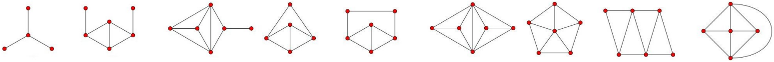

The bound does **not** hold for claw-free graphs.

Question

Does the bound hold for a class defined by a smaller subfamily of the list?

Theorem (F., Gravier, Naserasr, Parreau, Valicov)

Let G be an edge-identifiable graph with a minimal edge identifying code C_E. Then $G[C_E]$ is 2-degenerated.

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This is almost tight since $\gamma^{\text{EID}}(K_{2,n}) = 2n - 2 = 2|V(K_{2,n})| - 6$.

An upper bound - corollary

Corollary

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Corollary

If G is an edge-identifiable graph with average degree $\overline{d}(G) \geq 5$, then $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq n - \frac{n}{\Delta(\mathcal{L}(G))}$ where $n = |V(\mathcal{L}(G))|$.

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Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected identifiable graph on n vertices and of maximum degree Δ. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + O(1)$.

Problem EDGE IDCODE

INSTANCE: A graph G and an integer k.

QUESTION: Does G have an edge identifying code of size at most k?

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Theorem (F., Gravier, Naserasr, Parreau, Valicov)

EDGE IDCODE is NP-complete, even for planar subcubic bipartite graphs of arbitrarily large girth.

Proof by reduction from:

Problem PLANAR ($\leq 3, 3$)-SAT

INSTANCE: A set $\mathcal{Q}$ of clauses over a set X of boolean variables such that:

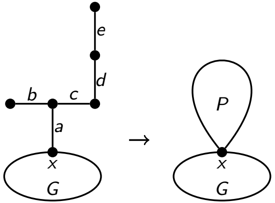
- Each clause contains at least two and at most three distinct literals
- Each variable appears exactly once negated, twice non-negated
- The bipartite incidence graph $B(\mathcal{Q})$ is planar

QUESTION: Can $\mathcal{Q}$ be satisfied, *i.e.* is there a truth assignment of the variables of X such that each clause contains at least one true literal?

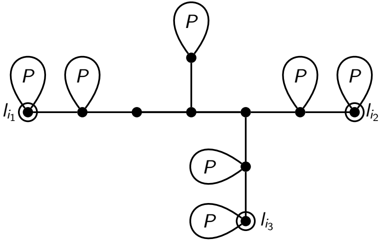
Theorem (Dahlhaus, Johnson, Papadimitriou, Seymour, Yannakakis, 1994)

PLANAR ($\leq 3, 3$)-SAT is NP-complete.

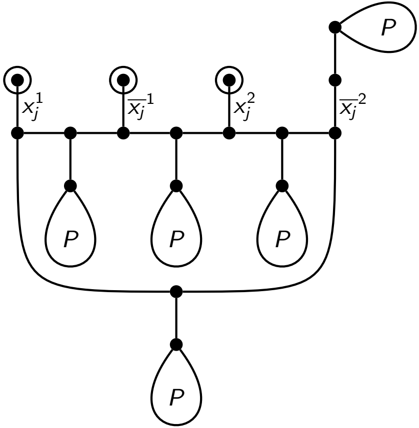
Reduction



Clause gadget



Variable gadget



$\mathcal{Q}$ is satisfiable if and only if G contains an edge identifying code C_E of size $k = 25|\mathcal{Q}| + 22|X|$.

Theorem (Trotter, 1977)

A line graph $\mathcal{L}(G)$ is perfect if and only if G has no odd cycles of length more than 3

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Corollary

IDCODE is NP-complete even when restricted to perfect 3-colorable planar line graphs of maximum degree 4.

Theorem (Courcelle, 1990)

Every graph property expressable in monadic second-order logic is solvable in linear time in classes of graphs having bounded tree-width.

Corollary

EDGE IDCODE is linear time sovable in trees, k-outerplanar graphs, series-parallel graphs, ...

Graph: set V of vertices, set E of edges, unary predicates $a, b : E \to V$

- $e \neq f := \big(a(e) \neq a(f) \wedge a(e) \neq b(f)\big) \vee \big(b(e) \neq a(f) \wedge b(e) \neq b(f)\big)$
- $e\mathcal{I}^* f := a(e) = a(f) \vee a(e) = b(f) \vee b(e) = b(f) \vee b(e) = a(f)$

$$\exists C, C \subseteq E, |C| \leq k, \big(\forall e \in E, \exists f \in C \wedge e\mathcal{I}^* f\big) \wedge$$
$$\Big(\forall e \in E, \forall f \in E, e \neq f, \exists g \in C, \big((e\mathcal{I}^* g \wedge \neg(f\mathcal{I}^* g)) \vee (f\mathcal{I}^* g \wedge \neg(e\mathcal{I}^* g))\big)\Big)$$

Gràcies!