Identifying codes in graphs

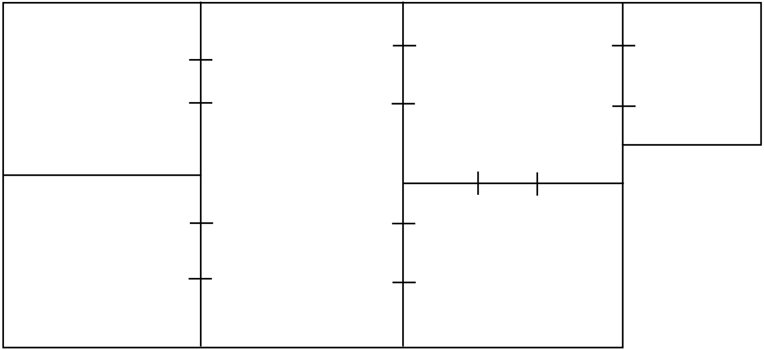
Problems from the other side of the Pyrenees

Florent Foucaud

Combgraph seminar

February 21st, 2013

Identifying the rooms of a building



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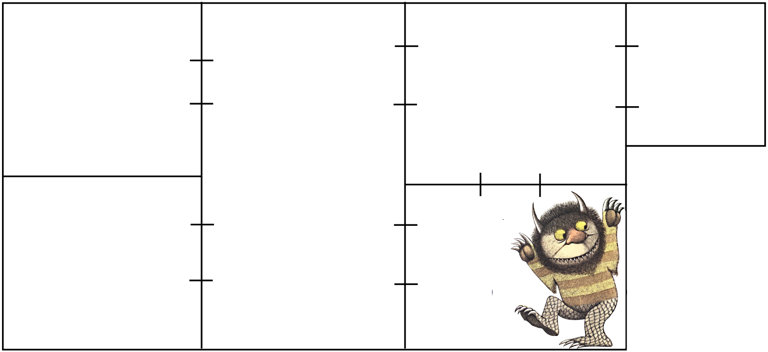

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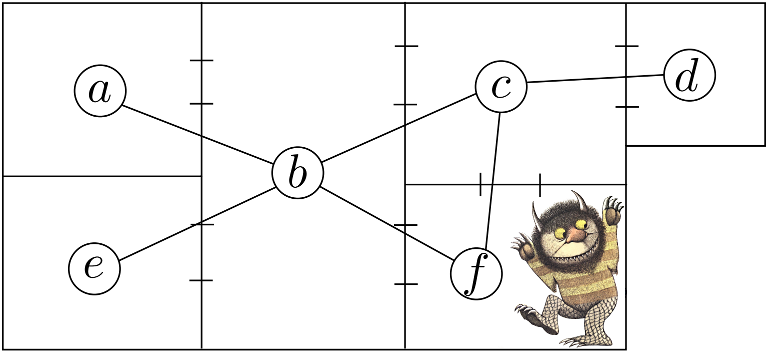

Graph $G = (V, E)$. V: vertices (rooms), $E \subseteq V \times V$: edges (doors)

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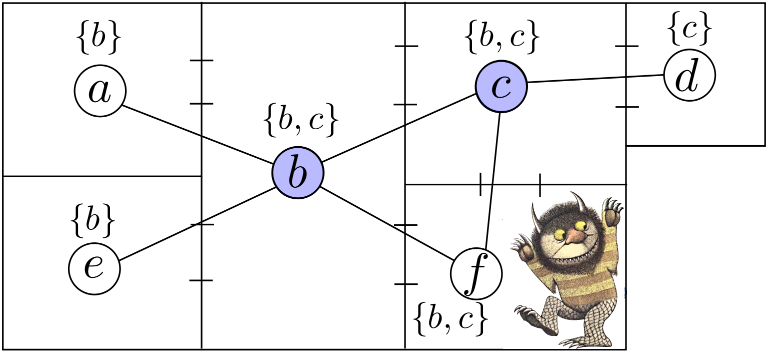

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Motion detector: detects intruder in its room or in adjacent rooms

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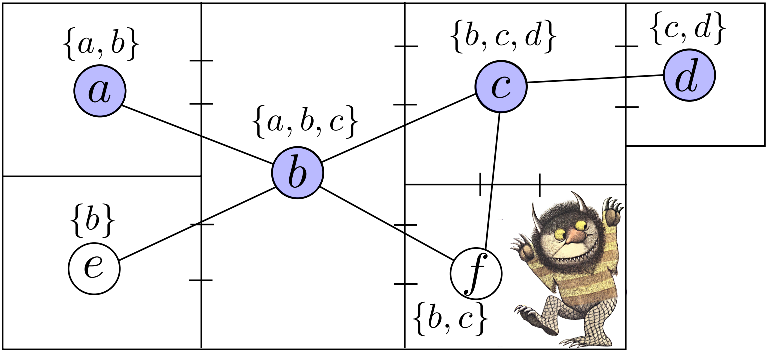

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Identifying codes

G: undirected graph
$N[u]$: set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

- C is a **dominating set**: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
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Goal: minimize number of detectors

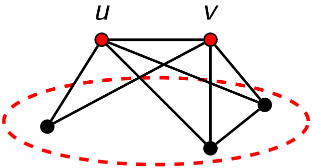
$\gamma^{\text{ID}}(G)$: minimum size of an identifying code in G

Identifiable graphs

Remark

Not all graphs have an identifying code!

Twins = pair u, v such that $N[u] = N[v]$.



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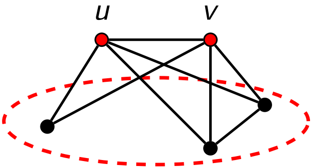

Proposition

A graph is **identifiable** if and only if it is **twin-free** (i.e. has no twins).

Bounds on $\gamma^{\text{ID}}(G)$

n: number of vertices

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

G identifiable graph on n vertices:

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G)$$

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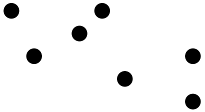
$\gamma^{\text{ID}}(G) = n \Leftrightarrow G$ has no edges



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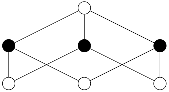

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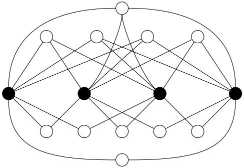

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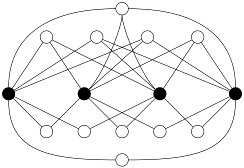

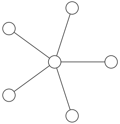

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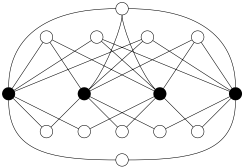

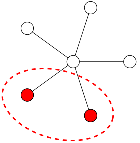

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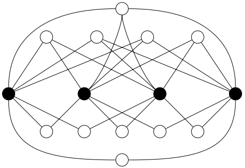

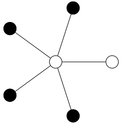

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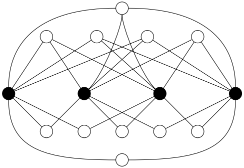

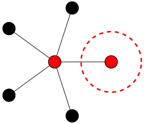

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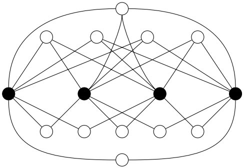

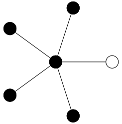

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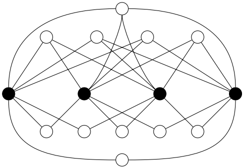

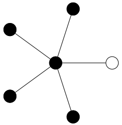

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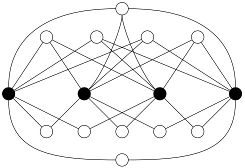

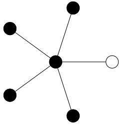

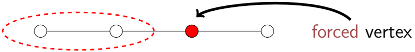

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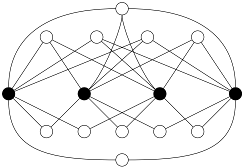
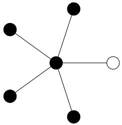
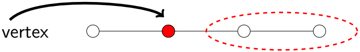




forced vertex



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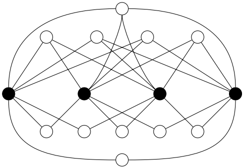

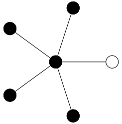

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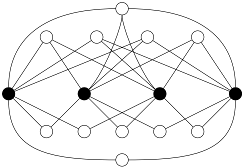
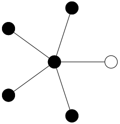




A question

Theorem (Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

G identifiable graph on n vertices with at least one edge:

$$\gamma^{\text{ID}}(G) \leq n - 1$$

Question

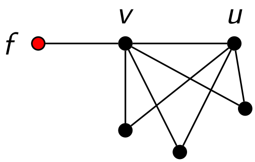
What are the graphs G with n vertices and $\gamma^{\text{ID}}(G) = n - 1$?

Forced vertices

u, v such that $N[v] \ominus N[u] = \{f\}$:

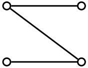
f belongs to **any identifying code**

$\rightarrow f$ forced by u, v.

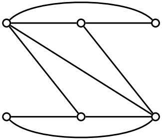


Graphs with many forced vertices

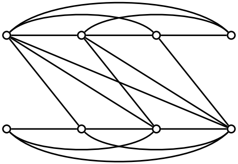
Special path powers: $A_k = P_{2k}^{k-1}$



$A_2 = P_4$



$A_3 = P_6^2$



$A_4 = P_8^3$

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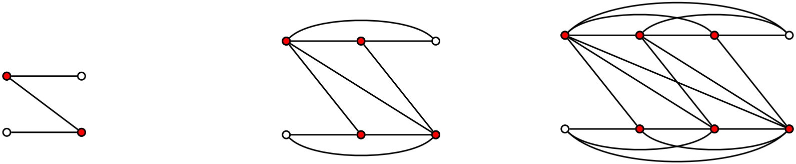

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Graphs with many forced vertices

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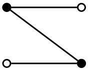
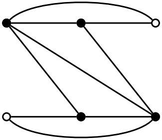
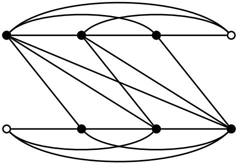


$A_2 = P_4$



$A_3 = P_6^2$



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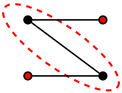

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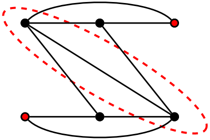

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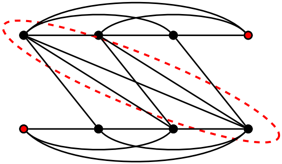

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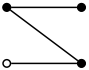

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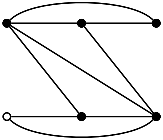

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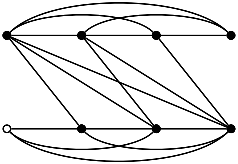
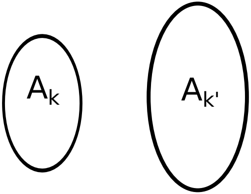


$A_4 = P_8^3$

Proposition

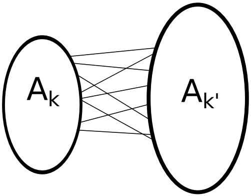
$$\gamma^{\mathrm{ID}}(A_k) = n - 1$$

Constructions using joins



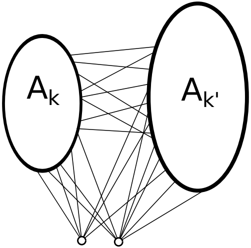
Two graphs A_k and $A_{k'}$

Constructions using joins



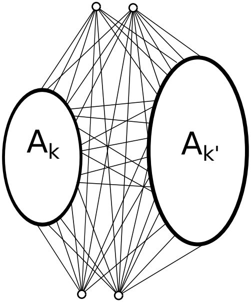
Join: add all edges between them

Constructions using joins



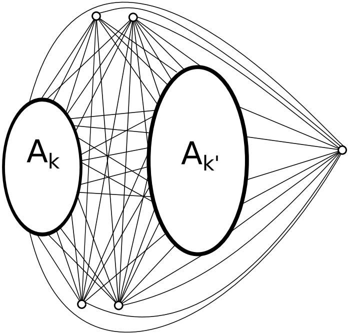
Join the new graph to two non-adjacent vertices ($\overline{K_2}$)

Constructions using joins



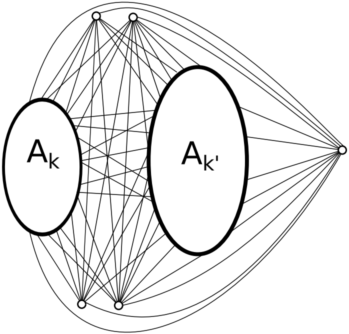
Join the new graph to two non-adjacent vertices, again

Constructions using joins



Finally, add a **universal vertex**

Constructions using joins



Finally, add a **universal vertex**

Proposition

At each step, the constructed graph has $\gamma^{\text{ID}} = n - 1$

A characterization

(1) stars

(2) $A_k = P_{2k}^{k-1}$

(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$

(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$\gamma^{\text{ID}}(G) = n - 1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

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Observation

All these graphs have maximum degree $n-1$ or $n-2$

The maximum degree

A lower bound using the maximum degree

maximum degree of G: maximum number of neighbours of a vertex in G

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

G identifiable graph, n vertices, maximum degree Δ:

$$\frac{2n}{\Delta+2} \leq \gamma^{\text{ID}}(G)$$

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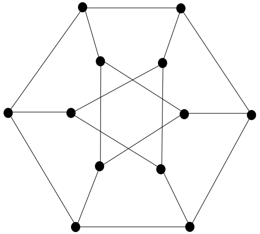
Equality if and only if G can be constructed as follows:

- Take Δ-regular graph H



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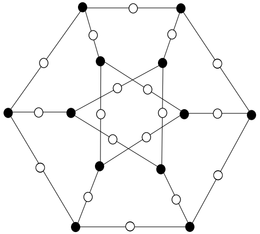

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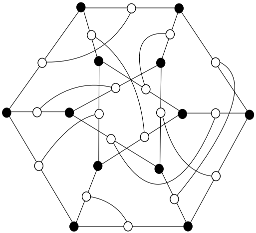
- Take Δ-regular graph H
- Subdivide each edge once
- Possibly add some edges



The influence of the maximum degree

Question

What is a good **upper bound** on γ^{ID} using the maximum degree?

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Proposition

There exist graphs with n vertices, max. degree Δ and $\gamma^{\text{ID}}(G) = n - \frac{n}{\Delta}$.

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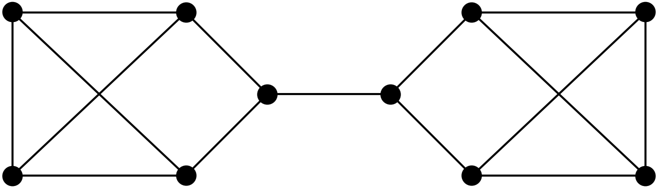

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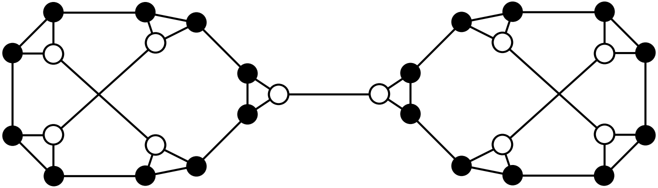

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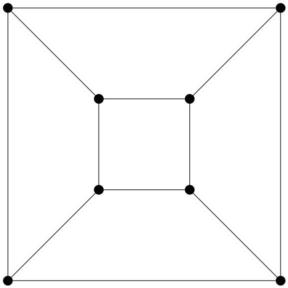

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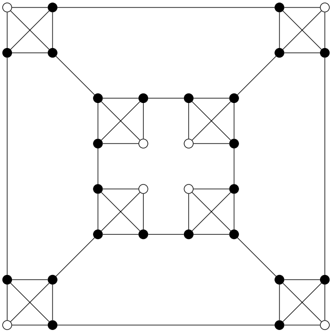

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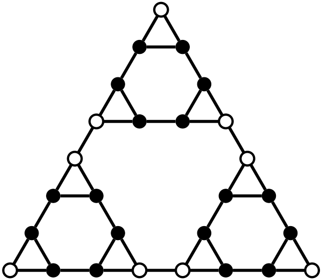
Also: Sierpiński graphs

(Gravier, Kovše, Mollard, Moncel, Parreau, 2011)



A conjecture

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

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Question

Can we prove that $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta)}$?

Triangle-free graphs

Theorem (F., Klasing, Kosowski, Raspaud, 2009)

G identifiable triangle-free graph, n vertices, max. degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta + \frac{3\Delta}{\ln \Delta - 1}} = n - \frac{n}{\Delta(1 + o_\Delta(1))}$$

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Proof idea: Constructive.

Triangle-free graphs have **large** independent sets

(see e.g. Shearer: $\alpha(G) \geq \frac{\ln \Delta}{\Delta} n$)

$\rightarrow$ Locally modify such an independent set:

its complement is a "small" id. code.

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Remark

Same technique applies to families of triangle-free graphs with large independent sets.

$\rightarrow$ bipartite graphs: $\alpha(G) \geq \frac{n}{2} \Rightarrow \gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+9}$

Upper bounds for $\gamma^{\text{ID}}(G)$

Theorem (F., Perarnau, 2012)

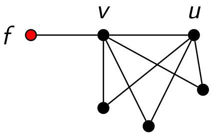
G identifiable graph, n vertices, maximum degree Δ, no isolated vertices:

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{105\Delta}$$

Notation

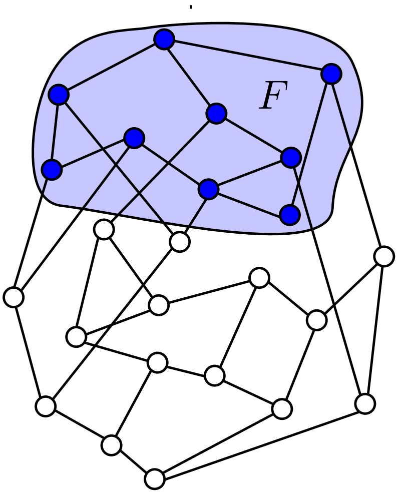
$NF(G)$: proportion of non forced vertices of G

$$NF(G) = \frac{\#\text{non forced vertices in G}}{\#\text{vertices in G}}$$



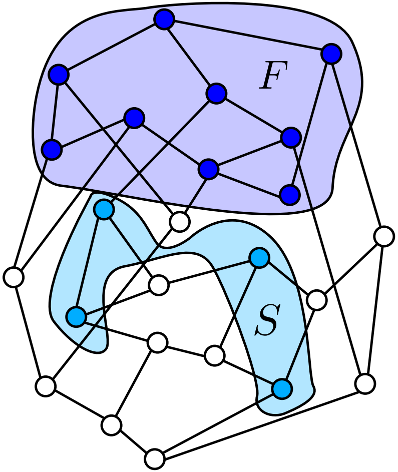
Proof

F: forced vertices.



Proof

F: forced vertices. Select "big" **random set** S from $V(G) \setminus F$



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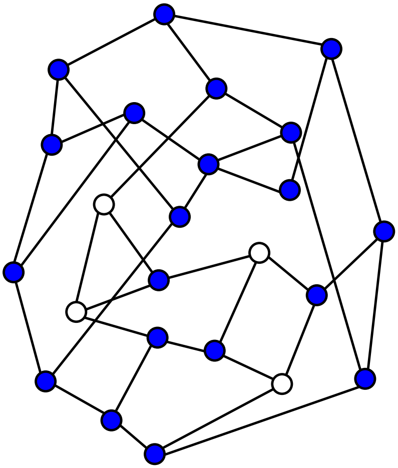
Goal: $\mathcal{C} = V(G) \setminus S$ small identifying code



Want:
$\mathbb{E}(|S|) = p \cdot nNF(G) = \frac{nNF(G)}{\Theta(\Delta)}$

$$\mathbb{E}(|\mathcal{C}|) = n - \frac{nNF(G)}{\Theta(\Delta)}$$

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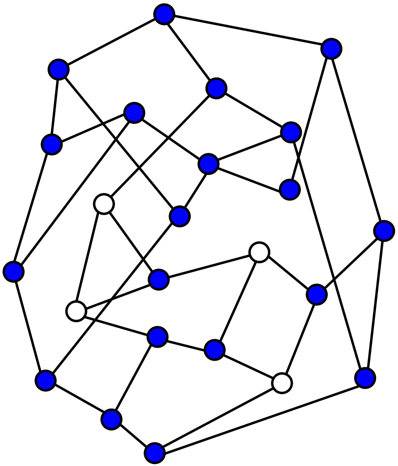

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Apply Lovász Local Lemma + Chernoff bound on S

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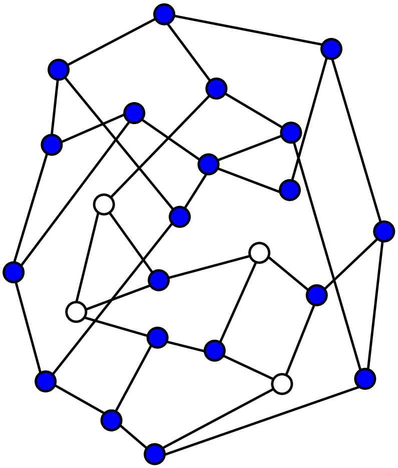

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Apply Lovász Local Lemma + Chernoff bound on S

with positive prob. $|S|$ is close to expected size, and we are done.

Bounding the number of forced vertices

$NF(G)$: proportion of non forced vertices of G

Theorem (F., Perarnau, 2012)

G identifiable graph on n vertices having maximum degree Δ and no isolated vertices:

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{105\Delta}$$

Question

What can be said about $NF(G)$?

Bounding the number of forced vertices

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Question

What can be said about $NF(G)$?

$$G \text{ regular} \Rightarrow NF(G) = 1$$

Corollary

G regular: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{105\Delta}$

Bounding the number of forced vertices

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Lemma (Bertrand, 2005)

G: identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex in $N[x]$.
$\rightarrow$ Set of non forced vertices is a **dominating set**.

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Corollary

$$\frac{1}{\Delta+1} \leq NF(G) \leq 1 \text{ and } \gamma^{\text{ID}}(G) \leq n - \frac{n}{105(\Delta+1)^3}$$

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clique number of G: max. size of a complete subgraph in G

Proposition (F., Perarnau, 2012)

Let G be a graph of clique number at most k. There exists a (huge) function c such that:

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Summary

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

G connected identifiable graph, n vertices, max. degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \tfrac{n}{\Delta} + c \text{ for some constant } c$$

Theorem

in general: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$

triangle-free: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta(1+o_\Delta(1))}$

bipartite: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+9}$

no forced vertices (e.g. regular): $\gamma^{\text{ID}}(G) \leq n - \frac{n}{105\Delta}$

clique number k: $n - \frac{n}{105c(k)^2\Delta}$

line graph of a graph H with $\overline{d}(H) \geq 5$: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta}$

Open questions

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

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Question

Can we prove the conjecture, or at least $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta)}$? for, e.g.:

- $\Delta = 3$?
- trees?
- **all** line graphs?
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Question

How to handle forced vertices?

The minimum degree

Graphs with girth at least 5

Proposition (F., Perarnau, 2012)

G twin-free graph, n vertices, girth at least 5. D, 2-dominating set of G. If $G[D]$ has no isolated edge, D is an identifying code.

Graphs with girth at least 5

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Theorem (F., Perarnau, 2012)

G twin-free graph, girth at least 5, min. degree δ. Then

$$\gamma^{\text{ID}}(G) \leq \frac{3\left(\ln\delta + \ln\ln\delta + 1 + \frac{\ln\ln\delta}{\ln\delta} + \frac{1}{\ln\delta}\right)}{2\delta} = (1 + o_\delta(1))\frac{3\ln\delta}{2\delta}n$$

If $\overline{d}(G) = O_\delta(\delta(\ln\delta)^2)$ (in particular, when G regular) then

$$\gamma^{\text{ID}}(G) \leq \frac{\ln\delta + \ln\ln\delta + O_\delta(1)}{\delta}n$$

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$$\gamma^{\text{ID}}(G) \leq \frac{\ln\delta + \ln\ln\delta + O_\delta(1)}{\delta}n$$

Corollary

G random d-regular graph. Then a.a.s.

$$\gamma^{\text{ID}}(G) \leq \frac{\log d + \log\log d + O_d(1)}{d}n$$

Sketch of the proof: construct 2-dominating set D

Proof similar as random construction of domination set
(Alon and Spencer, Chapter 1: Alteration method)

- $S \subseteq V$ at random, each element with probability p.

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- $S \subseteq V$ at random, each element with probability p.
- $Pr(v \text{ not 2-dom. }) = (1-p)^{d+1} + (d+1)p(1-p)^d \leq (1+dp)e^{-dp}$

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(Alon and Spencer, Chapter 1: Alteration method)

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- $Pr(v \text{ not 2-dom. }) = (1-p)^{d+1} + (d+1)p(1-p)^d \leq (1+dp)e^{-dp}$
- $X(S) =$ non 2-dominated vertices

Sketch of the proof: construct 2-dominating set D

Proof similar as random construction of domination set (Alon and Spencer, Chapter 1: Alteration method)

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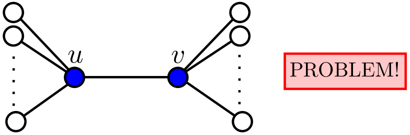
- $\mathcal{C} = S \cup \{v : v \in X(S)\}$, $p = \frac{\log d + \log\log d}{d}$

$$\mathbb{E}(|D|) = \mathbb{E}(|S|) + |X(S)| \leq \frac{\log d + \log\log d}{d} n + \frac{1 + \log d + \log\log d}{d \log d} n$$

Sketch of the proof: identifying code



$Pr(\text{isolated edge}) \leq p^2(1-p)^{2d-2} + (1-p)^{2d} + p(1-p)^{2d-1}$ **SMALL**

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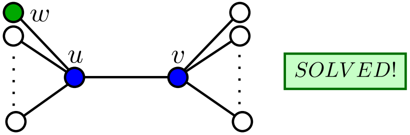

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$\mathcal{C} = S \cup \{v : v \in X(S)\} \cup \{w : w \in N(u),\ uv \text{ isolated edge}\}$,
$p = \frac{\log d + \log\log d}{d}$

$$\mathbb{E}(|\mathcal{C}|) \leq \frac{\log d + \log\log d + O_d(1)}{d} n$$

Minimum degree 2

Theorem (F., Klasing, Kosowski, 2009)

G twin-free graph, n vertices, minimum degree at least 2, girth at least 5. Then $\gamma^{\text{ID}}(G) \leq \frac{7n}{8}$.

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Proof idea: Build DFS-spanning tree

Take three out of four levels.

Possibly add $\leq \frac{n}{8}$ vertices to fix conflicts.

Comparison with dominating sets

$\gamma(G)$: domination number of G

Theorem (Payan, 60's - easy proof in Alon and Spencer's book)

G, n vertices, min. degree δ. Then $\gamma(G) \leq \frac{1+\ln(\delta+1)}{\delta+1}n$.

Theorem

G, n vertices. All bounds are tight.

- min. degree 1: $\gamma(G) \leq \frac{n}{2}$ (Folklore)
- connected, min. degree 2: $\gamma(G) \leq \frac{2n}{5}$ except for 7 small graphs (McCuaig-Shepherd, 1989)
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Question

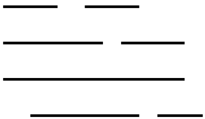
Can we prove similar bounds for γ^{ID} and girth 5 ?

Interval and line graphs

Interval graphs

Theorem (F., Naserasr, Parreau, Valicov, 2012+)

G interval graph: $\gamma^{\text{ID}}(G) > \sqrt{2n}$



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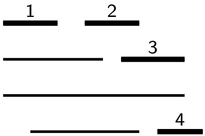

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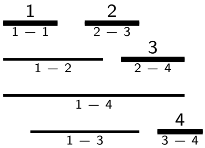

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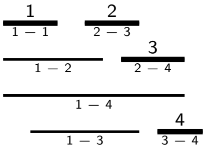

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$$\rightarrow n \leq \sum_{i=1}^{k} i = \binom{k}{2}$$

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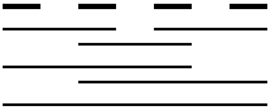
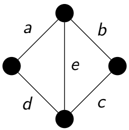
Tight



Line graphs

Definition - Line graph of H: Edge-adjacency graph of H

Denoted $\mathcal{L}(H)$
$V(\mathcal{L}(H)) = E(H)$
$e \sim e'$ in $\mathcal{L}(H)$ iff e and e' incident to common vertex in H



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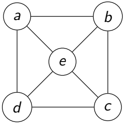

$\mathcal{L}(H)$

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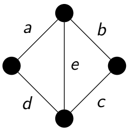

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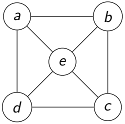
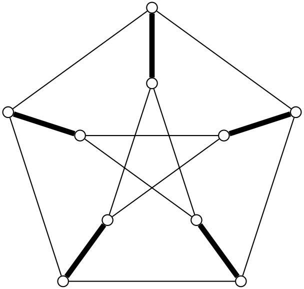


$\mathcal{L}(H)$

Tool: edge-identifying codes

Edge-identifying code of $H \iff$ Identifying code of $\mathcal{L}(H)$

Edge-identifying code - example



$$\gamma^{\text{EID}}(\mathcal{P}) \leq 5$$

A lower bound for line graphs

Theorem (F., Gravier, Naserasr, Parreau, Valicov, 2012)

$$\gamma^{\text{ID}}(\mathcal{L}(H)) = \gamma^{\text{EID}}(H) \geq \frac{|V(H)|}{2}$$

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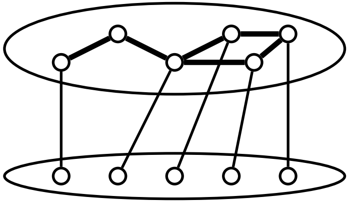
Proof idea:

C_E, k edges on n' vertices

$X = V(G) \setminus V(C_E)$



- Assume C_E is connected
- If C_E has a cycle, $|X| \leq n' \leq k$,
- If C_E is a tree, $n' - 1 = k$ and $|X| \leq n' - 2$
- In both cases, $n = |X| + n' \leq 2k$

A lower bound for line graphs

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$$\gamma^{\text{ID}}(\mathcal{L}(H)) = \gamma^{\text{EID}}(H) \geq \frac{|V(H)|}{2}$$

Since $|V(\mathcal{L}(H))| = |E(H)| \leq \frac{|V(H)|(|V(H)|-1)}{2}$

Corollary

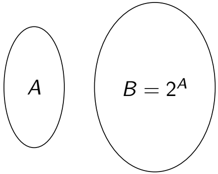
$$\gamma^{\text{ID}}(\mathcal{L}(H)) \geq \frac{\sqrt{2|V(\mathcal{L}(H))|}}{2}$$

No extension to quasi-line graphs!

$A = \{a_1, \ldots, a_k\}$, $B = 2^A$: cliques.

$|V(G)| = k + 2^k$

$\gamma^{\text{ID}}(G) \leq 2k = \Theta(\log(|V(G)|))$



Open questions

Bounds in $\Omega(\sqrt{n})$ for interval and line graphs.

Question

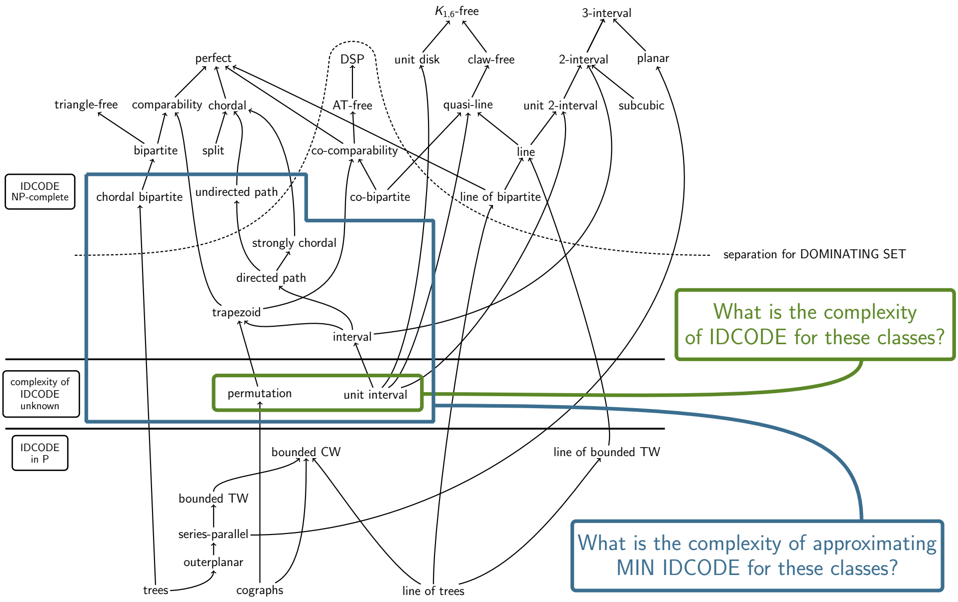
Is there some common point between these two results?

Question

What about other nice classes, e.g. permutation graphs?

Computational problems

Complexity of (MIN) IDCODE for various graph classes



Conclusion

Open problems

- Better upper bound on γ^{ID} depending on Δ. Conjecture: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + c$

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- Computational aspects of identifying codes