Identifying codes in graphs of given maximum degree

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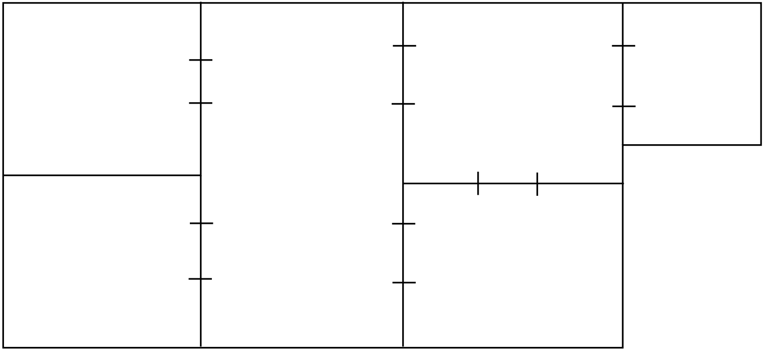
joint works with:

Ralf Klasing, Adrian Kosowski, André Raspaud (2012)
Eleonora Guerrini, Matjaž Kovše, Aline Parreau, Reza Naserasr, Petru Valicov (2011)
Guillem Perarnau (2012)
Sylvain Gravier, Aline Parreau, Reza Naserasr, Petru Valicov (2012)

LRI, Paris, 25.05.2012

Locating a burglar in a museum



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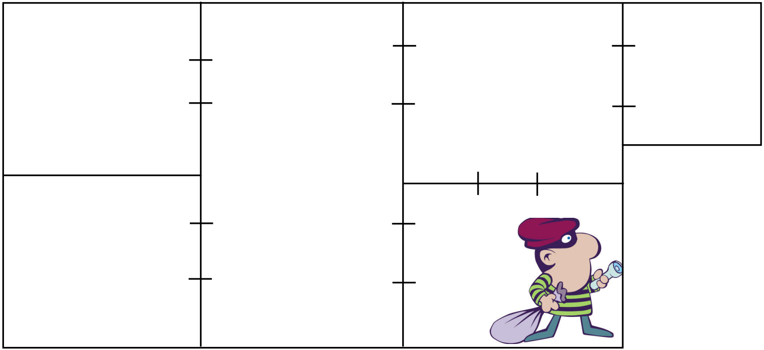
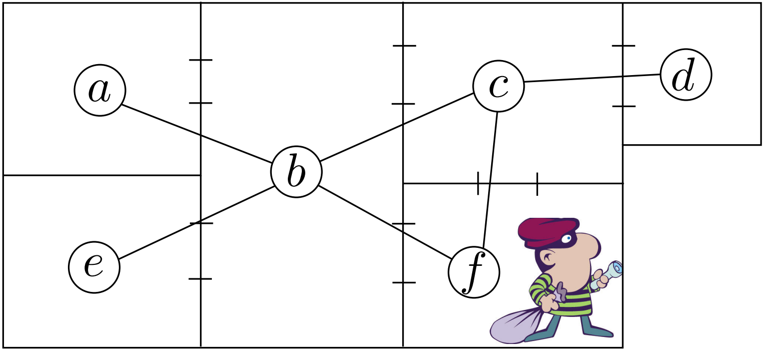
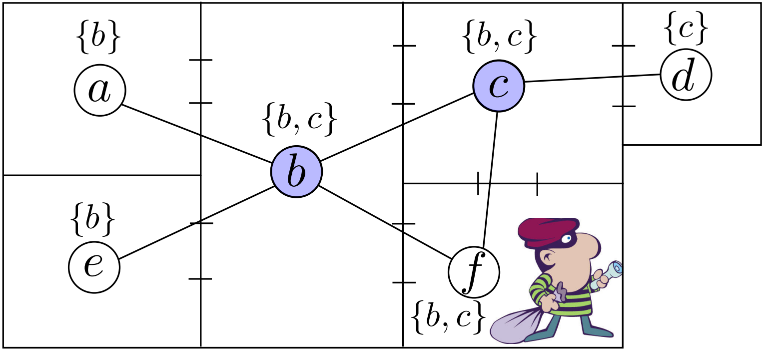




Graph $G = (V, E)$. V: vertices (rooms), $E \subseteq V \times V$: edges (doors)



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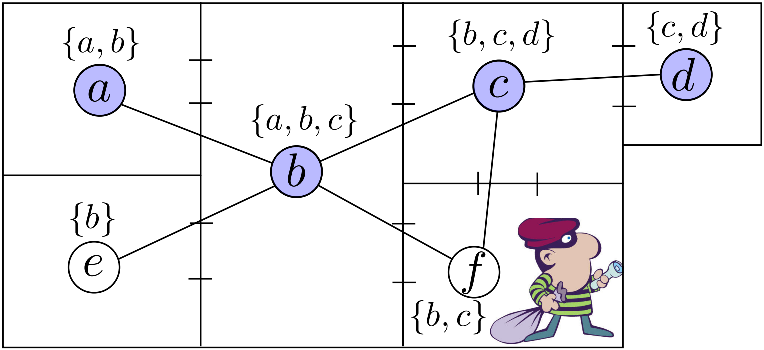

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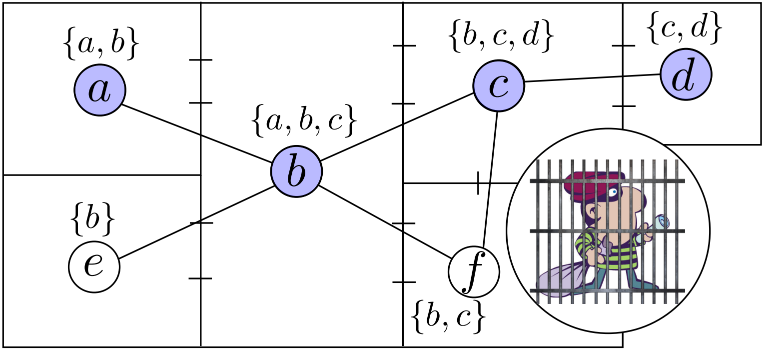

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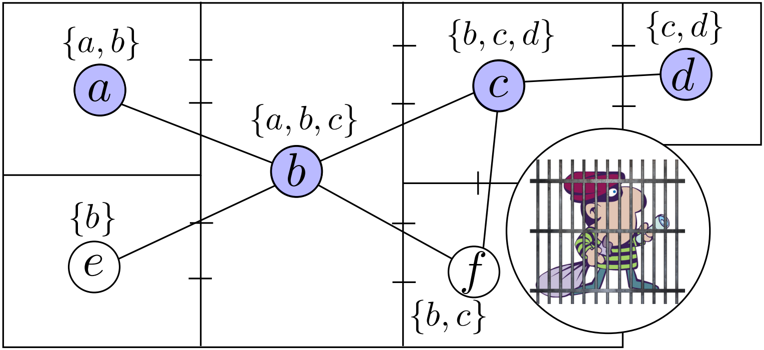

How many detectors do we need?

Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a **dominating set** in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a **separating code** in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

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Proposition

C is an identifying code IFF:

- C is a **dominating set** in G
- $\forall u \neq v$ of V **with** $d_G(u,v) \leq 2$, $(N[u] \Delta N[v]) \cap C \neq \emptyset$

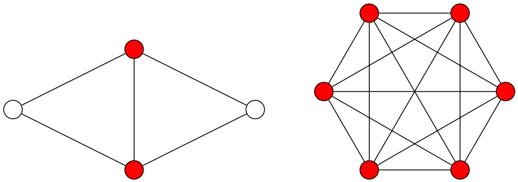
$N[u]$: set of vertices v s.t. $d(u, v) \leq 1$

Remark

Not all graphs have an identifying code!

Twins = pair u, v such that $N[u] = N[v]$.

A graph is **identifiable** iff it is **twin-free** (i.e. it has no twins).



The test cover problem

Remark

Identifying codes can be seen as a special case of the **test cover problem** (a.k.a. test collection problem).

Example on board.

Bounds not related to $\Delta(G)$

Theorem (lower bound: Karpovsky, Chakrabarty, Levitin, 1998
upper bound: Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

Let G be an identifiable graph on n vertices with at least one edge, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

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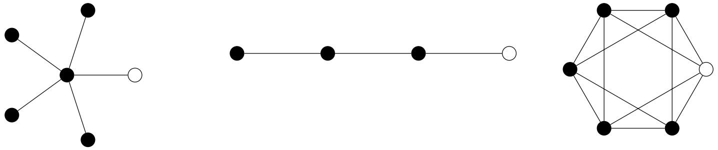
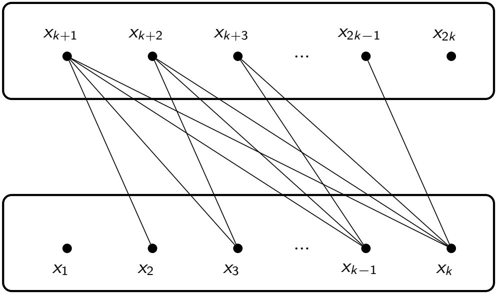


Definition - Graph A_k

$V(A_k) = \{x_1, ..., x_{2k}\}$.
x_i connected to x_j iff $|j - i| \leq k - 1$

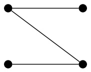
Note: $A_1 = \overline{K_2}$; for $k \geq 2$, $A_k = P_{2k}^{k-1}$



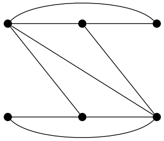
Clique on $\{x_{k+1}, ..., x_{2k}\}$

Clique on $\{x_1, ..., x_k\}$

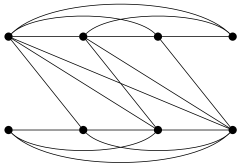
$A_1 = \overline{K_2}$



$A_2 = P_4$



$A_3 = P_6^2$



$A_4 = P_8^3$

Definition - Join and its closure

$(\mathcal{A}, \bowtie)$: closure of graphs of $\mathcal{A}$ with respect to $\bowtie$ (complete join).

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

Let G be an identifiable graph on n vertices. Then:

$$\gamma^{\text{ID}}(G) = n - 1 \Leftrightarrow G \in \{K_{1,n-1}\} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1 \text{ and } G \neq \overline{K_2}.$$

A characterization

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Observation

All these graphs have maximum degree $n - 1$ or $n - 2$!

A conjecture

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

Let G be an identifiable graph with maximum degree Δ and n vertices, then

$$\frac{2n}{\Delta+2} \leq \gamma^{\text{ID}}(G)$$

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Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected nontrivial identifiable graph on n vertices and of maximum degree Δ. Then:

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + c \text{ (for some constant } c\text{).}$$

The conjecture is true for $\Delta = 2$ (with $c = 3/2$).

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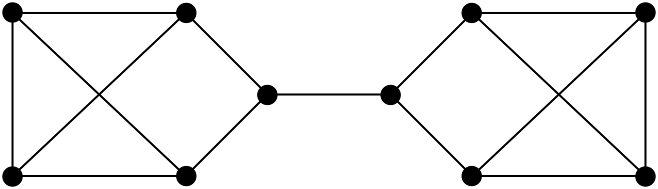

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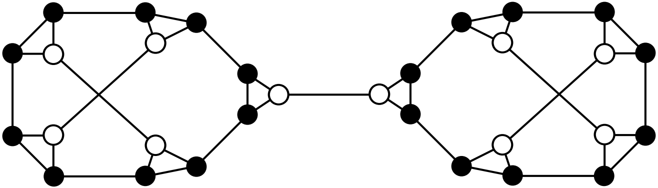

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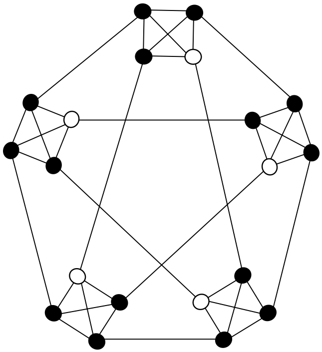

Extremal examples

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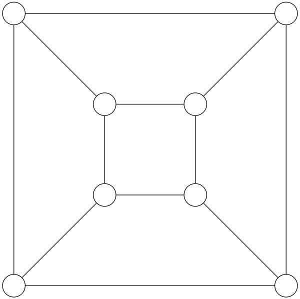

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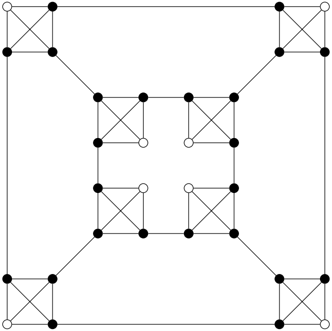

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Also: Sierpiński graphs
(see A. Parreau, S. Gravier, M. Kovše, M. Mollard and J. Moncel, 2011+)

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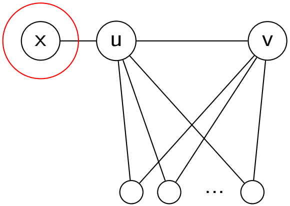
Question

Can we prove that $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta)}$?

u, v such that $N[v] \ominus N[u] = \{x\}$

Then $x \in C$, forced by uv.



Note: if G regular, no forced vertices.

Theorem (F., Guerrini, Kovse, Naserasr, Parreau, Valicov, 2011)

Let G be a connected identifiable graph of maximum degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^5)}$$

If G is Δ-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$

Proof idea:

Proposition

Let I be a distance 4-independent set of G. If for all $x \in I$, x is **not forced**, $V - I$ is also an identifying code.

First bounds

Theorem (F., Guerrini, Kovse, Naserasr, Parreau, Valicov, 2011)

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Lemma (Bertrand, Hudry, 2001)

For each vertex x of G, there exists a non forced vertex y in $N[x]$.

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Lemma (Bertrand, Hudry, 2001)

For each vertex x of G, there exists a non forced vertex y in $N[x]$.

Take a (maximal) 6-independent set I. Find the set I' "good vertices" which are not forced: $|I| = |I'|$. $V - I'$ is an identifying code.
For regular graphs, there are no forced vertices: a 4-IS is enough.

Theorem (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected identifiable **triangle-free** graph on n vertices and of maximum degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{(1+\frac{3}{\ln\Delta - 1})\Delta} = n - \frac{n}{(1+o_\Delta(1))\Delta}$$

Proof idea:

Let X be the set of vertices having at least some **false twin** (false twins: $u \not\sim v$ and $N(u) = N(v)$).

- If X is large, at least $\frac{|X|}{\Delta}$ vertices can be out of a code and we are done
- Otherwise, build a maximal independent set S with $|S| > \frac{\ln\Delta}{\Delta}n$ (using J. Shearer's bound)
- Locally modify S to get S', not too small: $|S'| \geq |S|/3$
- $V \setminus S'$ is an identifying code

Triangle-free graphs

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Let G be a connected identifiable **triangle-free** graph on n vertices and of maximum degree Δ. Then

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Theorem (F., Klasing, Kosowski, Raspaud, 2009)

In fact: let G be a connected identifiable **triangle-free** graph on n vertices and of maximum degree Δ s.t. for all subgraphs H, $\alpha(H) \geq f(\Delta)n_H$. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta + \frac{3}{f(\Delta)}}$$

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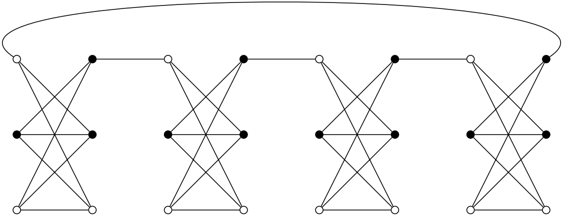
Corollary

G k-colourable: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+3k}$.
$\Rightarrow$ Bipartite: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+6}$
$\Rightarrow$ Planar triangle-free: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+9}$

Triangle-free graphs - examples

Complete $(\Delta - 1)$-ary tree, caterpillar: roughly, $\gamma^{\text{ID}}(G) = n - \frac{n}{\Delta - 1}$



$\gamma^{\text{ID}}(G) = n - \frac{n}{2\Delta/3}$

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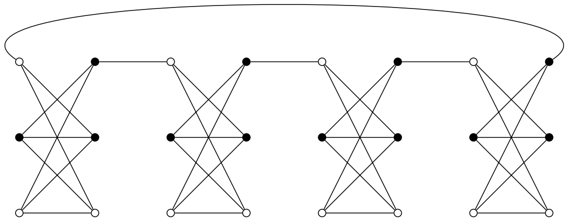

$\gamma^{\text{ID}}(G) = n - \frac{n}{2\Delta/3}$

Theorem (F., Klasing, Kosowski, Raspaud, 2009)

In fact: let G be a connected identifiable **triangle-free** graph on n vertices and of maximum degree Δ and **without false twins**. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\frac{3\Delta}{\ln \Delta - 1}} = n - \frac{n}{o(\Delta)}$$

So, any counterexample or extremal example should have false twins.

Notation

Let $NF(G)$ be the proportion of **non** forced vertices of G

$$NF(G) = \frac{\#\text{non-forced vertices in G}}{\#\text{vertices in G}}$$

Theorem (F., Perarnau, 2011)

For each identifiable graph G on n vertices having maximum degree $\Delta \geq 3$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{103\Delta}$$

Proof idea:

- Take all forced vertices (set F) into the code.
- From $V \setminus F$, select each vertex with probability $p_S = \frac{1}{k \cdot \Delta}$ (k constant) to belong to a set S. We want $C = V \setminus S$.
- Use Lovász' Local Lemma to show that $\Pr(C \text{ is a code}) > f(k, n, \Delta) > 0$
- Use the Chernoff bound to show that $\Pr(C \text{ is too small}) < f(k, n, \Delta)$

Bounding the number of forced vertices

Proposition

$$\frac{1}{\Delta+1} \leq NF(G) \leq 1$$

Proof:

Lemma (Bertrand, Hudry, 2001)

Let G be an identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex y in $N[x]$.

$\Rightarrow$ The set S of non-forced vertices forms a dominating set. Hence $|S| \geq \frac{n}{\Delta+1}$.

Bounding the number of forced vertices

Proposition

Let G be a graph of clique number at most k. There exists a function ρ such that:

$$\frac{1}{\rho(k)} \leq NF(G) \leq 1$$

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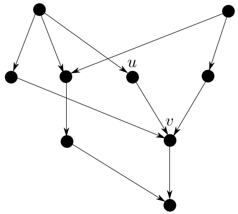
$$\frac{1}{\rho(k)} \leq NF(G) \leq 1$$

- Define graph $\overrightarrow{H}(G)$
- Max. degree of $\overrightarrow{H}(G)$: $2k-3$
- Longest directed chain of $\overrightarrow{H}(G)$: $k-1$
- Each component has a non-forced vertex
- $\Rightarrow \rho(k) \leq \sum_{i=0}^{k-2}(2k-3)^i$



$u \rightarrow v$ if $N[v] = N[u] \cup \{x\}$

Corollaries

Theorem (F., Perarnau, 2011)

For each identifiable graph G on n vertices having maximum degree $\Delta \geq 3$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{103\Delta}$$

Corollary

- In general, $NF(G) \geq \frac{1}{\Delta+1}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$
- If G is Δ-regular, $NF(G) = 1$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{103\Delta} = n - \frac{n}{\Theta(\Delta)}$
- If G has clique number bounded by k, $NF(G) \geq \frac{1}{\rho(k)}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{103 \cdot (\rho(k))^2 \cdot \Delta} = n - \frac{n}{\Theta(\Delta)}$

Note: for $k = 2, 3, 4, 5$: $103 \cdot (\rho(k))^2 = 103$, 1.360, 81.685, 13.600.000

The conjecture holds for some large subclass of line graphs:

Theorem (F., Gravier, Naserasr, Parreau, Valicov, 2011)

Let G be an edge-identifiable graph with a minimal edge-identifying code C_E. Then $G[C_E]$ is 2-degenerate.

Corollary

If G edge-identifiable, $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq 2|V(G)| - 3$.

Corollary

If G is an edge-identifiable graph **with average degree** $\overline{d}(G) \geq 5$, then $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq n - \frac{n}{\Delta(\mathcal{L}(G))}$ where $n = |V(\mathcal{L}(G))|$.

Questions

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected nontrivial identifiable graph on n vertices and of maximum degree Δ. Then:

$$\gamma^{\mathrm{ID}}(G) \leq n - \frac{n}{\Delta} + c \text{ (for some constant } c\text{)}.$$

- Can we reduce the constants?
- Can we improve the bound $n - \frac{n}{\Theta(\Delta^3)}$?
- What about $\Delta = 3$?
- What about trees (having a look at David Auger's algorithm)?
- What about claw-free graphs? $n - \frac{n}{\Theta(\Delta^2)}$ seems to hold by directly using similar arguments than for triangle-free graphs.
- Other related parameters?