Identification problems in graphs

Florent Foucaud (Univ. Blaise Pascal, Clermont-Ferrand, France)

joint works with:

Eleonora Guerrini, Mike Henning, Matjaž Kovše,
Christian Löwenstein, George B. Mertzios, Reza Naserasr,
Aline Parreau, Thomas Sasse, Petru Valicov

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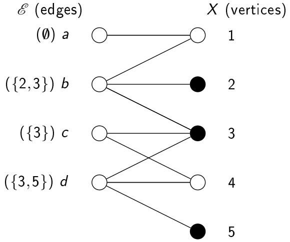
Identification problems

Definition - Separating system (Rényi, 1961)

Hypergraph $(X, \mathscr{E})$. Find subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

also knwn as Distinguishing set, Test cover, Distinguishing transversal, Discriminating code...



example: $C = \{2,3,5\}$

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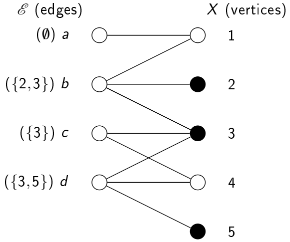

example: $C = \{2, 3, 5\}$

Remark

Equivalently: for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f

Theorem (Folklore)

For set system $(X, \mathscr{E})$, a separating system has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C.
Hence $|\mathscr{E}| \leq 2^{|\mathscr{C}|}$. □

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Hence $|\mathscr{E}| \leq 2^{|\mathscr{C}|}$. □

Theorem (Bondy's theorem, 1972)

For set system $(X, \mathscr{E})$, a minimal separating system has size at most $|\mathscr{E}| - 1$.

Proof: nice and short graph-theoretic argument. □

Identifying codes in graphs

G: undirected graph
$N[u]$: set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

- C is a **dominating set**: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
- C is a **separating code**: $\forall u \neq v$ of $V(G)$, $N[u] \cap C \neq N[v] \cap C$

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$ID(G)$: identifying code number of G,
minimum size of an identifying code in G

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Domination number: $DOM(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Examples: paths

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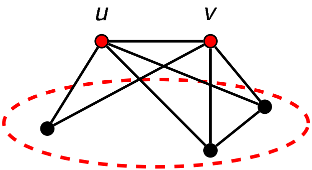
Domination number: $DOM(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Identifying code number: $ID(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$

Remark

Not all graphs have an identifying code!

Closed twins = pair u, v such that $N[u] = N[v]$.



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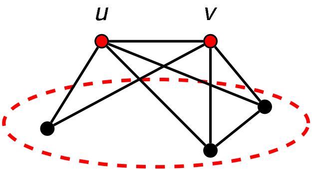

Proposition

A graph is **identifiable** if and only if it is **closed twin-free** (i.e. has no twins).

n: number of vertices

Theorem (Folklore)

G identifiable graph on n vertices:

$$\lceil \log_2(n+1) \rceil \leq ID(G)$$

Bounds on $ID(G)$

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G identifiable graph on n vertices with at least one edge:

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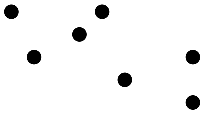
$ID(G) = n \Leftrightarrow G$ has no edges



Further examples

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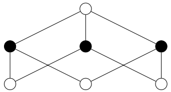
$ID(G) = \log_2(n+1)$



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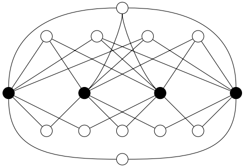

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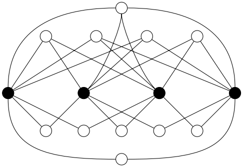

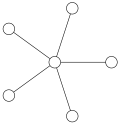

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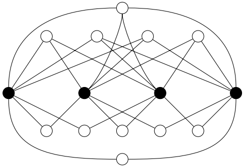

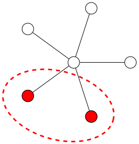

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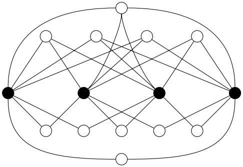

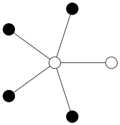

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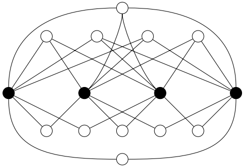

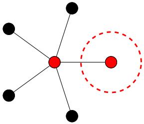

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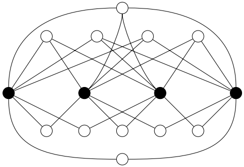

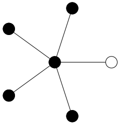

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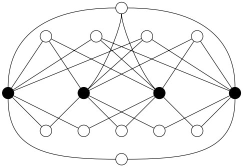

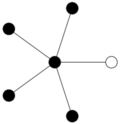

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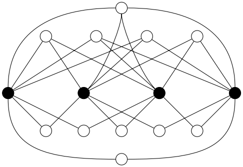

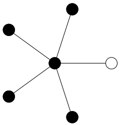

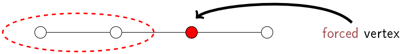

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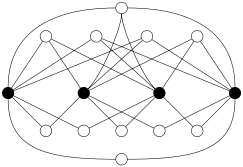

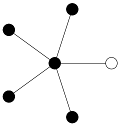

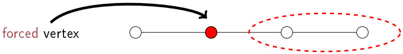

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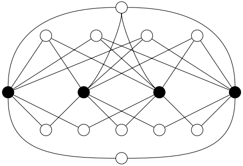

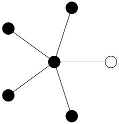

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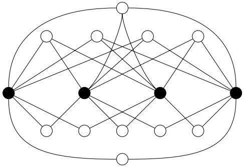
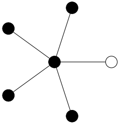




Theorem (Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

G identifiable graph on n vertices with at least one edge:

$$ID(G) \leq n-1$$

Question

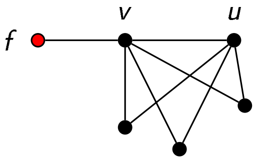
What are the graphs G with n vertices and $ID(G) = n-1$?

Forced vertices

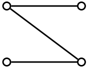
u, v such that $N[v] \ominus N[u] = \{f\}$:

f belongs to **any identifying code**

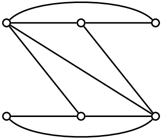
$\rightarrow f$ forced by u, v.



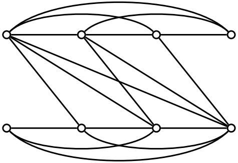
Special path powers: $A_k = P_{2k}^{k-1}$



$A_2 = P_4$



$A_3 = P_6^2$



$A_4 = P_8^3$

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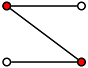

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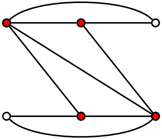

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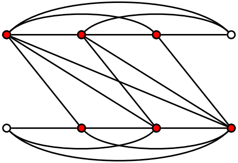
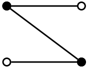
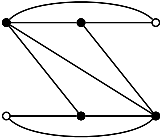


$A_4 = P_8^3$

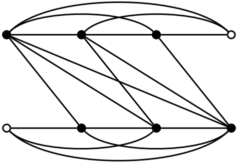
Special path powers: $A_k = P_{2k}^{k-1}$



$A_2 = P_4$



$A_3 = P_6^2$



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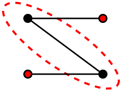

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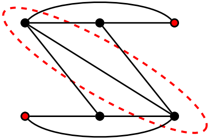

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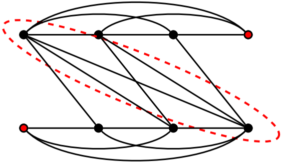

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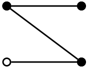

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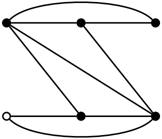

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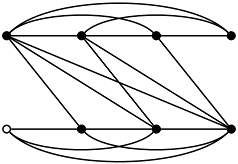
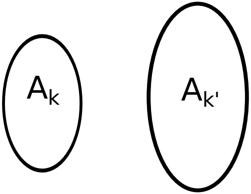


$A_4 = P_8^3$

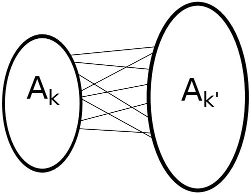
Proposition

$$ID(A_k) = n - 1$$



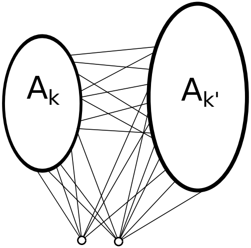
Two graphs A_k and $A_{k'}$

Constructions using joins

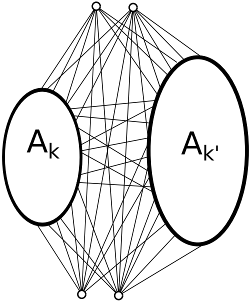


Join: add all edges between them

Constructions using joins

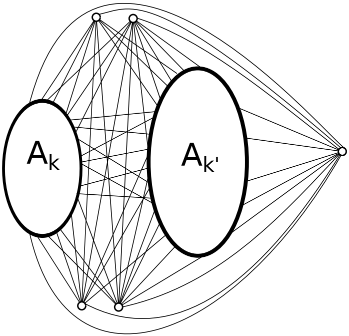


Join the new graph to two non-adjacent vertices ($\overline{K_2}$)



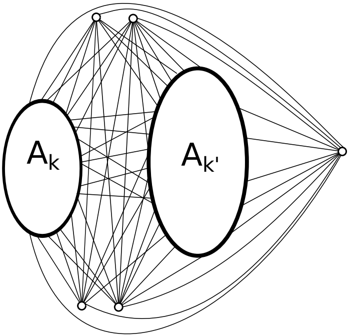
Join the new graph to two non-adjacent vertices, again

Constructions using joins



Finally, add a **universal vertex**

Constructions using joins



Finally, add a **universal vertex**

Proposition

At each step, the constructed graph has $ID = n - 1$

A characterization

(1) stars

(2) $A_k = P_{2k}^{k-1}$

(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$

(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$ID(G) = n - 1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

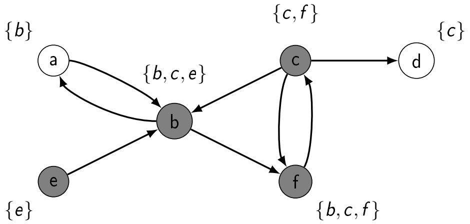
Identifying codes in digraphs

$N^-[u]$: *in-neighbourhood* of u

Definition - Identifying code of a digraph $D=(V,A)$

subset C of V such that:

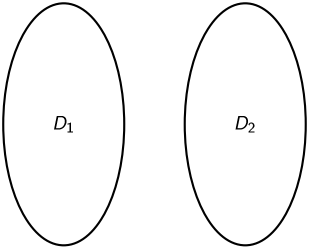
- C is a **dominating set** in D: for all $u \in V$, $N^-[u] \cap C \neq \emptyset$, and
- C is a **separating code** in D: for all $u \neq v$, $N^-[u] \cap C \neq N^-[v] \cap C$



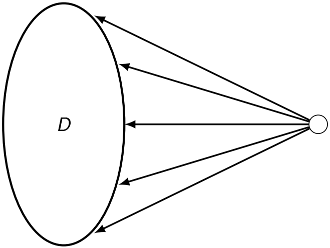
$ID(D)$: minimum size of an identifying code of D

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\triangleleft}(D)$: D joined to K_1 by incoming arcs only



$D_1 \oplus D_2$



$\overrightarrow{\triangleleft}(D)$

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\triangleleft}(D)$: D joined to K_1 by incoming arcs only

Definition

Let $(K_1, \oplus, \overrightarrow{\triangleleft})$ be the digraphs which can be built from K_1 by successive applications of $\oplus$ and $\overrightarrow{\triangleleft}$, starting with K_1.

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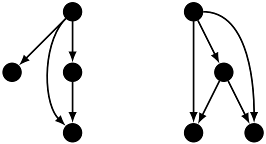

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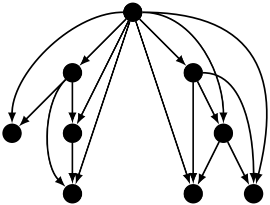

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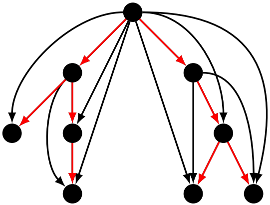
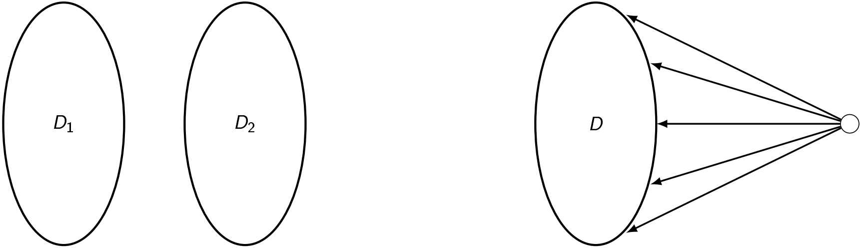


Proposition

For each digraph D of order n in $(K_1, \oplus, \overrightarrow{\triangleleft})$, $ID(D) = n$.

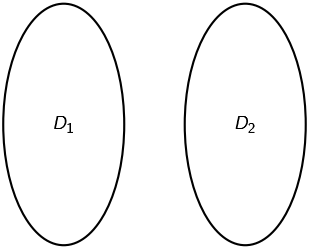


$D_1 \oplus D_2$

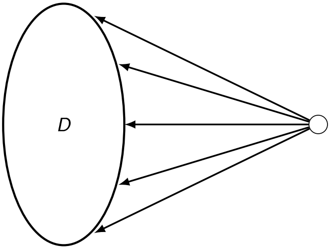
$\overrightarrow{\triangleleft}(D)$

Theorem (F., Naserasr, Parreau, 2013)

Let D be an identifiable digraph on n vertices. $ID(G) = n$ iff $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$.



$D_1 \oplus D_2$



$\overrightarrow{\triangleleft}(D)$

Location-domination in graphs

Definition - Locating-dominating set (Slater, 1980's)

subset D of vertices of $G=(V,E)$ which is:

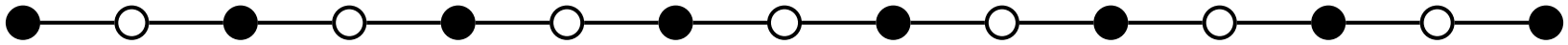
- **dominating** : $\forall u \in V, N[u] \cap D \neq \emptyset$,
- **locating** : $\forall u,v \in V \setminus D, N[u] \cap D \neq N[v] \cap D$.

$LD(G)$: location-domination number of G,
minimum size of a locating-dominating set of G.

Examples: paths

Domination number: $DOM(P_n) = \lceil \frac{n}{3} \rceil$

Identifying code number: $ID(P_n) = \lceil \frac{n+1}{2} \rceil$



Domination number: $DOM(P_n) = \lceil \frac{n}{3} \rceil$

Identifying code number: $ID(P_n) = \lceil \frac{n+1}{2} \rceil$

Location-domination number: $LD(P_n) = \lceil \frac{2n}{5} \rceil$

Theorem (Domination bound — Ore, 1960's)

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

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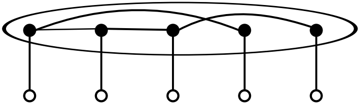
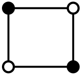
Tight examples:





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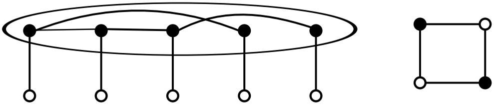
Tight examples: 

Theorem (Location-domination bound — Slater, 1980's)

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Theorem (Domination bound — Ore, 1960's)

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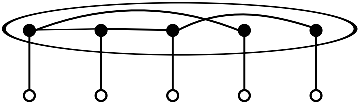
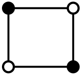
Tight examples:  

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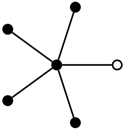
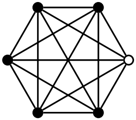
Tight examples:  

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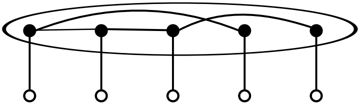
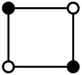
Tight examples:  

Theorem (Location-domination bound — Slater, 1980's)

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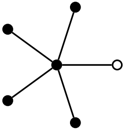
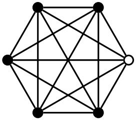
Tight examples:  

Remark: tight examples contain many twin-vertices!!

Upper bound - a conjecture

Theorem (Domination bound — Ore, 1960's)

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Theorem (Location-domination bound — Slater, 1980's)

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Theorem (Domination bound — Ore, 1960's)

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Theorem (Location-domination bound — Slater, 1980's)

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (Domination bound — Ore, 1960's)

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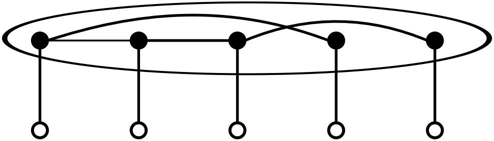
If true, tight: 1. domination-extremal graphs



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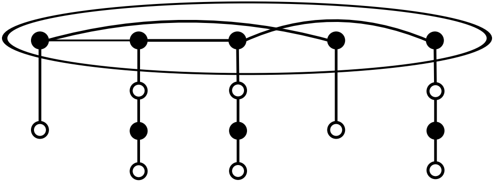
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 2. a similar construction



Upper bound - a conjecture

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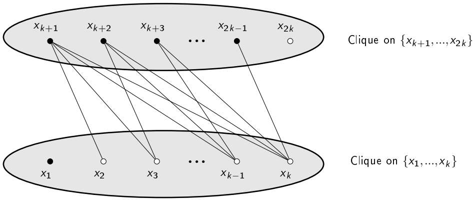
Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 3. a family with domination number 2



Upper bound - a conjecture

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Theorem (Garijo, González & Márquez, 2014)

Conjecture true if G has no 4-cycles, or if G is bipartite.

Upper bound - a conjecture

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (Garijo, González & Márquez, 2014)

Conjecture true if G has no 4-cycles, or if G is bipartite.

Proof ideas:

- no 4-cycles: use a maximum matching
- bipartite: every vertex cover is a locating-dominating set

Upper bound - a conjecture

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2014+)

Conjecture true if G is split graph or complement of bipartite graph.

Theorem (F., Henning, 2015+)

Conjecture true if G is:
- cubic graph
- line graph

Split graph: clique + independent set

Cubic graph: all degrees equal to 3

Line graph: Intersection graph of the edges of a graph

Upper bound - a conjecture

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2014+)

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Theorem (F., Henning, 2015+)

Conjecture true if G is:
- cubic graph
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Remark: Nontrivial proofs using very different techniques!
$\rightarrow$ Conjecture seems difficult.

Conjecture (Garijo, González & Márquez, 2014)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2014+)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

Lower bounds

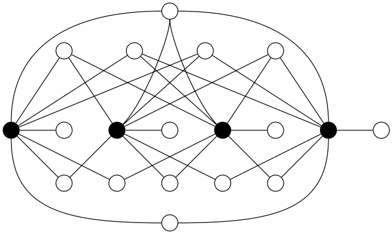
Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \rightarrow LD(G) = \Omega(\log n)$.

Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \rightarrow LD(G) = \Omega(\log n)$.

Tight example ($k = 4$):



Lower bounds

Theorem (Slater, 1980's)

G graph of order n, $LD(G)=k$. Then $n \leq 2^k + k - 1 \to LD(G) = \Omega(\log n)$.

Theorem (Slater, 1980's)

G tree of order n, $LD(G)=k$. Then $n \leq 3k-1 \to LD(G) \geq \frac{n+1}{3}$.

Theorem (Rall & Slater, 1980's)

G planar graph, order n, $LD(G)=k$. Then $n \leq 7k-10 \to LD(G) \geq \frac{n+10}{7}$.

Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$. Then $n \leq 2^k + k - 1 \rightarrow LD(G) = \Omega(\log n)$.

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Theorem (Rall & Slater, 1980's)

G planar graph, order n, $LD(G) = k$. Then $n \leq 7k - 10 \rightarrow LD(G) \geq \frac{n+10}{7}$.

Tight examples:

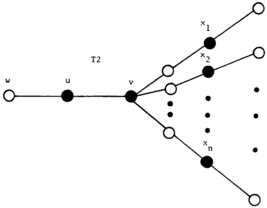


FIG. 2. Tree T2

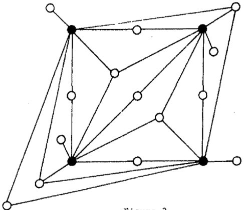
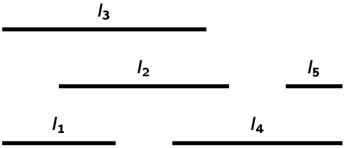
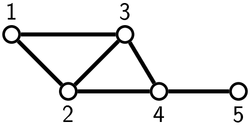


Figure 3.

Definition - Interval graph

Intersection graph of intervals of the real line.





Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

G interval graph of order n, $LD(G) = k$.

$$\text{Then } n \leq \frac{k(k+3)}{2}, \text{ i.e. } LD(G) = \Omega(\sqrt{n}).$$

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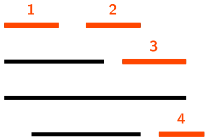

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- Define zones using the right points of intervals in D.

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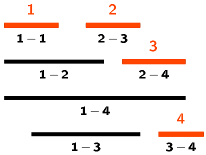

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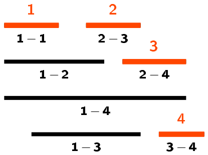

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- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

$\rightarrow n \leq \sum_{i=1}^{k}(k-i) + k = \frac{k(k+3)}{2}$.

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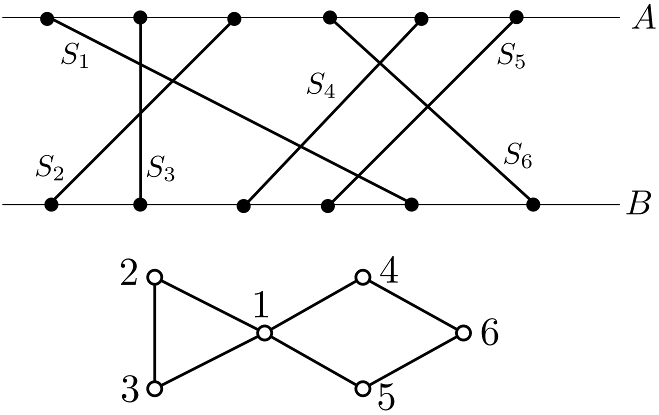
Tight:

Definition - Permutation graph

Given two parallel lines A and B:
intersection graph of segments joining A and B.



Lower bound for permutation graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

G permutation graph of order n, $LD(G)=k$.
Then $n \leq k^2+k-2$, i.e. $LD(G)=\Omega(\sqrt{n})$.

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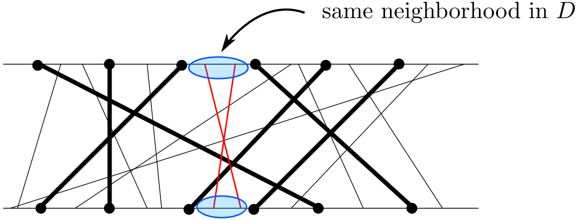

- Locating-sominating set D of size k: $k+1$ "top zones" and $k+1$ "bottom zones"
- Only one segment in $V \setminus D$ for one pair of zones

  $\rightarrow n \leq (k+1)^2 + k$
- Careful counting for the precise bound

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

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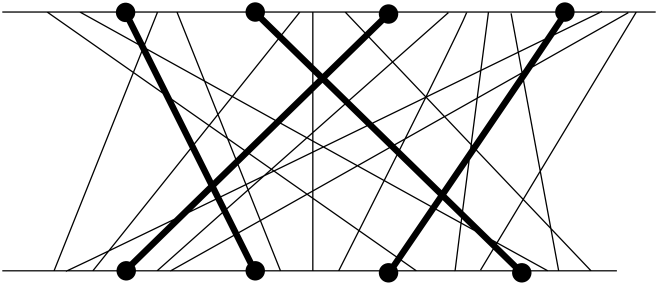
Tight:



Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

Let G be a graph on n vertices, $LD(G) = k$.

- If G is *unit* interval, then $n \leq 3k - 1$.
- If G is *bipartite* permutation, then $n \leq 3k + 2$.
- If G is a cograph, then $n \leq 3k$.

Set $X \subseteq V(G)$ is shattered:
for every subset $S \subseteq X$, there is a vertex v with $N[v] \cap X = S$

V-C dimension of G: maximum size of a shattered set in G

Vapnis-Chervonenkis dimension

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V-C dimension of G: maximum size of a shattered set in G

Theorem (Bousquet, Lagoutte, Li, Parreau, Thomassé, 2014+)

G graph of order n, $LD(G) = k$, V-C dimension $\leq d$. Then $n = O(k^d)$.

$\rightarrow$ interval graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

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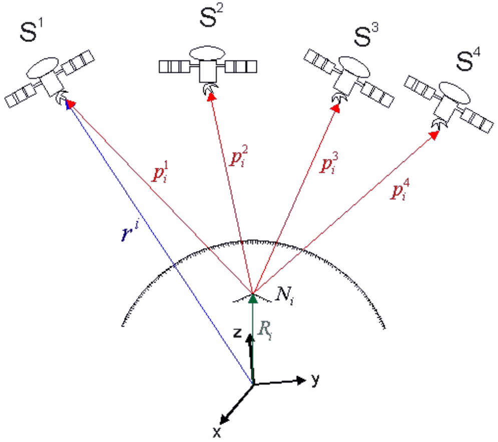
But better bounds exist:

- planar: $n \leq 7k - 10$ (Slater & Rall, 1984)
- line: $n \leq \frac{8}{9}k^2$ (F., Gravier, Naserasr, Parreau, Valicov, 2013)
- permutation: $n \leq O(k^2)$ (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

Metric dimension

Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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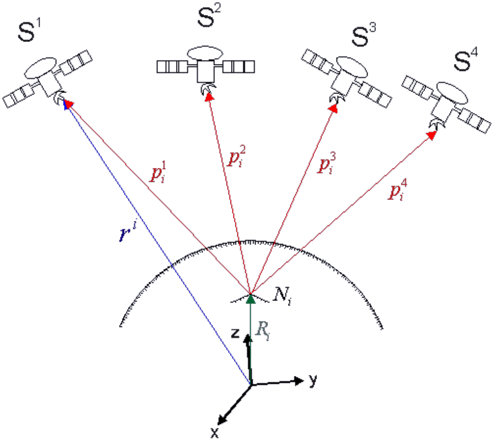

Question

Does the "GPS" approach also work in undirected unweighted graphs?

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)

$R \subseteq V(G)$ resolving set of G:

$\forall u \neq v$ in $V(G)$, there exists $w \in R$ that distinguishes $\{u, v\}$.

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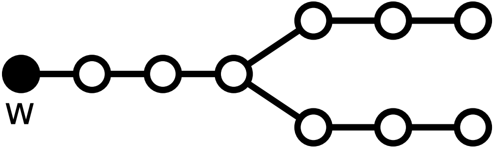

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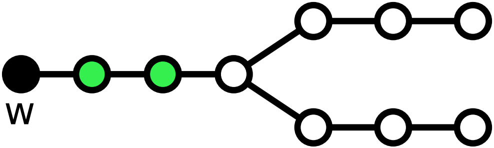

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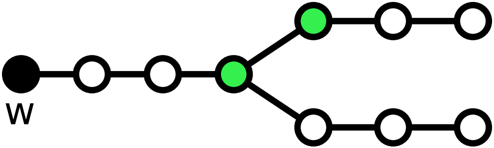

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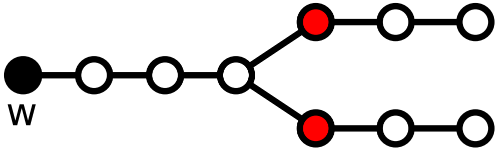

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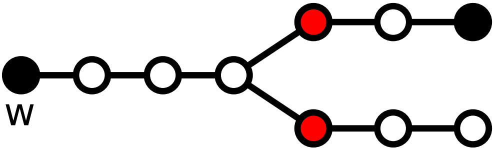

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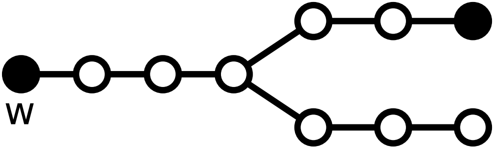

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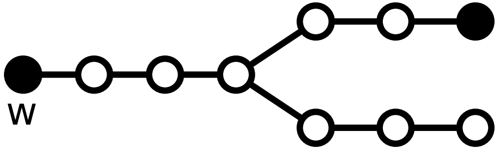

$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Remarks

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- Any locating-dominating set is a resolving set, hence $MD(G) \leq LD(G)$.
- A locating-dominating set can be seen as a "distance-1-resolving set".

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Proposition

$$MD(G) = 1 \Leftrightarrow G \text{ is a path}$$

Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

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Theorem (Khuller, Raghavachari & Rosenfeld, 2002)

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter: maximum distance between two vertices)

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→ Proofs are similar as for locating-dominating sets.

→ Bounds are tight (up to constant factors).

Complexity and algorithms

LOCATING-DOMINATING SET

INPUT: Graph G, integer k.
QUESTION: Is there a locating-dominating set of G of size k?

METRIC DIMENSION

INPUT: Graph G, integer k.
QUESTION: Is there a resolving set of G of size k?

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2015)

LOCATING-DOMINATING SET is NP-complete for graphs that are both interval and permutation.

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Reduction from 3-DIMENSIONAL MATCHING:

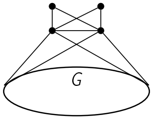
- INPUT: A, B, C sets and $\mathscr{T} \subset A \times B \times C$ triples
- QUESTION: is there a perfect 3-dimensional matching $M \subset T$, i.e., each element of $A \cup B \cup C$ appears exactly once in M?

Main idea: an interval can separate pairs of intervals **far away** from each other (without affecting what lies in between)

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2015)

LOCATING-DOMINATING SET is NP-complete for graphs that are both interval and permutation.

Reduction from LOCATING-DOMINATING SET to METRIC DIMENSION:

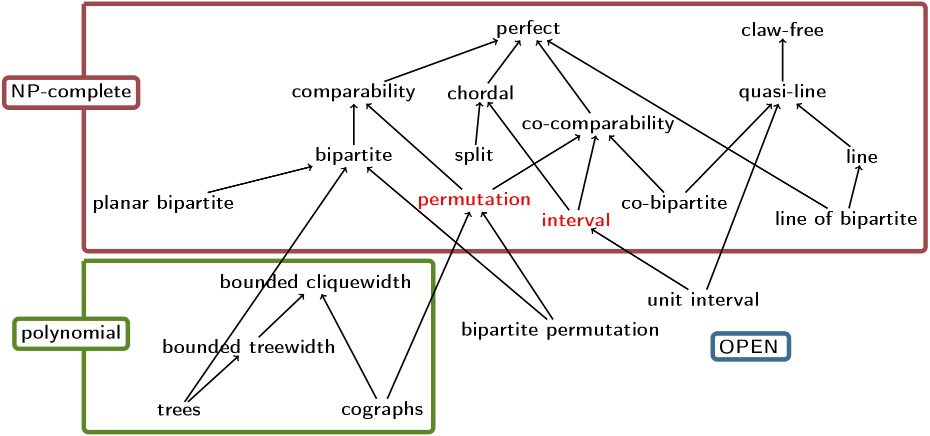


$MD(G') = LD(G) + 2$

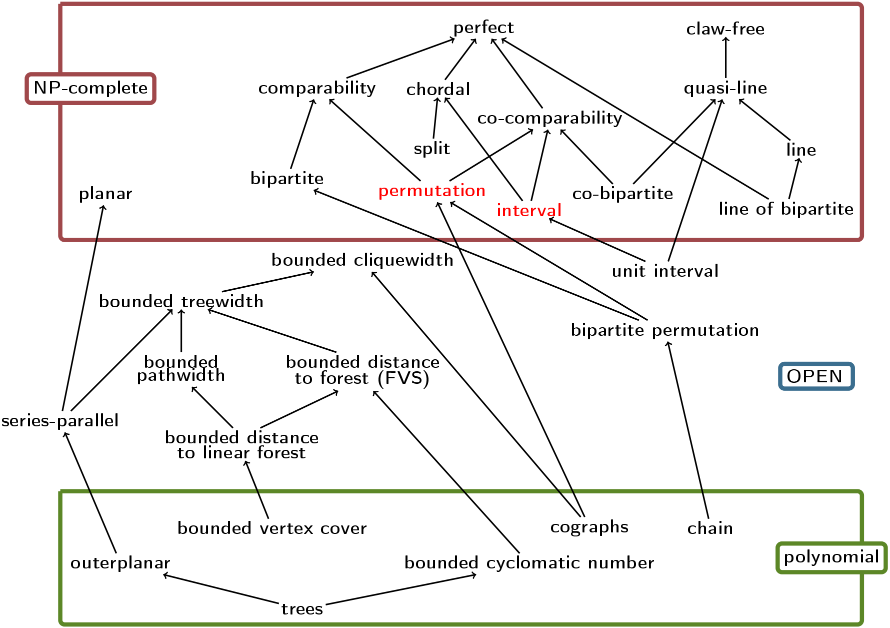
Corollary (F., Mertzios, Naserasr, Parreau, Valicov, 2015)

METRIC DIMENSION is NP-complete for graphs that are both interval and permutation.

Complexity of LOCATING-DOMINATING SET



Complexity of METRIC DIMENSION



An FPT algorithm for METRIC DIMENSION on interval graphs

Recall: METRIC DIMENSION W[2]-hard even for subcubic bipartite graphs
$\longrightarrow$ probably no $f(k)poly(n)$-time algorithm

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2015)

METRIC DIMENSION can be solved in time $2^{O(k^4)}n$ on interval graphs.

Ideas:

- use dynamic programming on a path-decomposition of G^4.
- each bag has size $O(k^2)$.
- it suffices to separate vertices at distance 2
- "transmission" lemma for separation constraints

To conclude

- Solve the conjecture: $LD(G) \leq \frac{n}{2}$ if G twin-free?
- Investigate bounds for other "geometric" graphs, for *MD* and *LD*
- Complexity of LOCATING-DOMINATING SET, METRIC DIMENSION on unit interval graphs
- Complexity of METRIC DIMENSION for bounded treewidth
- Parameterized complexity of METRIC DIMENSION: planar graphs, chordal graphs, permutation graphs...

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THANKS FOR YOUR ATTENTION