Bounds on the size of identifying codes for graphs of maximum degree Δ

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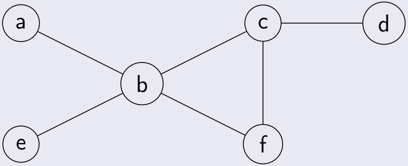
Locating a fire in a building

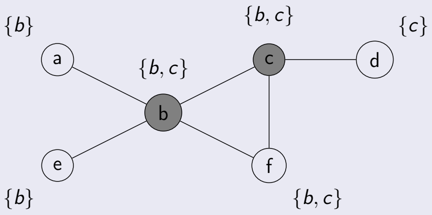
simple, undirected graph : models a building



Locating a fire in a building

simple detectors : able to detect a fire in a neighbouring room

goal : locate an eventual fire



		b	*c*	
a		•		
b		•	•	
c		•	•	
d			•	
e		•		
f		•	•	

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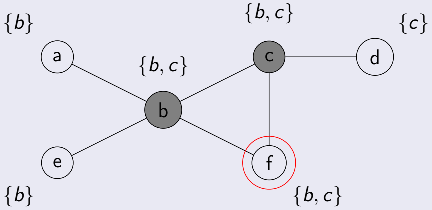

		b	c	
a		●		
b		●	●	
c		●	●	
d			●	
e		●		
f		●	●	

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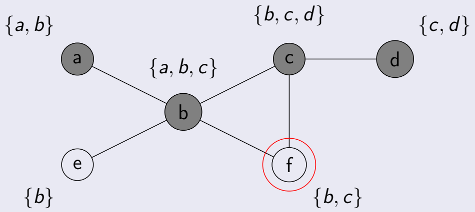
the *identifying sets* of all vertices must be distinct



	a	b	c	d
a	•	•		
b	•	•	•	
c		•	•	•
d			•	•
e		•		
f		•	•	

Identifying codes : definition

Definition : identifying code of a graph $G = (V, E)$ (Karpovsky et al. 1998 [4])

subset C of V such that :

- C is a dominating set in G, and
- for all distinct u, v of V, u and v have distinct *identifying sets* : $N[u] \cap C \neq N[v] \cap C$

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Remark

Note : close to *locating-dominating sets* (Slater, Rall 84 [6])

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Notation

$\gamma_{id}(G)$: minimum cardinality of an identifying code in a graph G

Identifiable graphs

Remark : not all graphs admit an identifying code

u and v are *twin* vertices if $N[u] = N[v]$.
A graph is *identifiable* iff it has no twin vertices.

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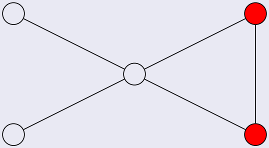
Non-identifiable graphs



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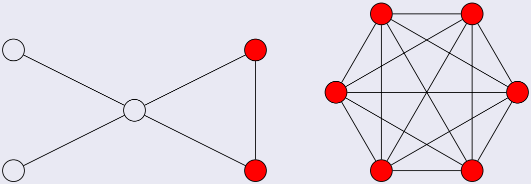

Hardness

Decision problem ID-CODE

Input : identifiable graph G and an integer k
Question : Is $\gamma_{id}(G) \leq k$?

Thm (Cohen et al. 01, Auger et al. 09)

ID-CODE is NP-complete even in bipartite and planar graphs.

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Optimization problem MIN ID-CODE

Input : identifiable graph G
Output : $\gamma_{id}(G)$

Thm (Trachtenberg et al. 06, Suomela 07, Gravier et al. 08 [1, 7, 2])

MIN ID-CODE can be approximated within a logarithmic factor, but not within a constant factor.

Lower bound and maximum degree

Thm (Karpovsky et al. 98 [4])

Let G be an identifiable graph with n vertices.
Then $\gamma_{id}(G) \geq \lceil \log_2(n+1) \rceil$.

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The graphs reaching this bound have been characterized (Moncel 06 [5])

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Thm (Karpovsky et al. 98 [4])

Let G be an identifiable graph with n vertices and maximum degree Δ.
Then $\gamma_{id}(G) \geq \dfrac{2n}{\Delta + 2}$.

Graphs reaching the lower bound

Characterization

- n vertices
- independent set C of size $\frac{2n}{\Delta+2}$ (id. code)
- every vertex of C has exactly Δ neighbours
- $\frac{\Delta n}{\Delta+2}$ vertices connected to exactly 2 code vertices each

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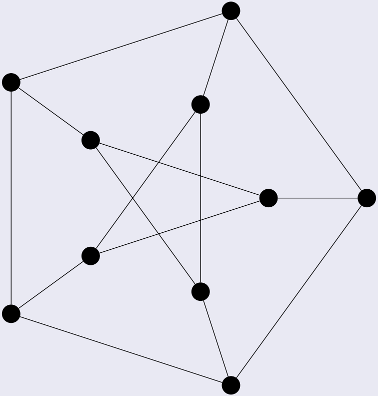
Construction

- Take a simple Δ-regular graph D (code)
- Put a new vertex on each edge of D
- Eventually add edges between the new vertices

Graphs reaching the lower bound - example

Example : D=Petersen graph, $\Delta = 3$, $n = 10$



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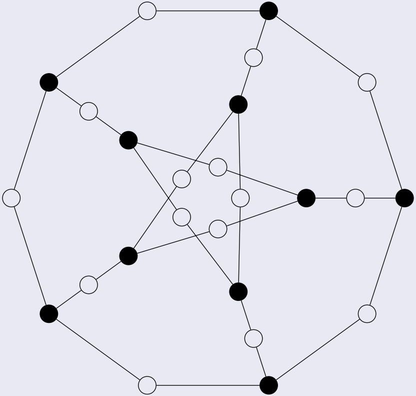

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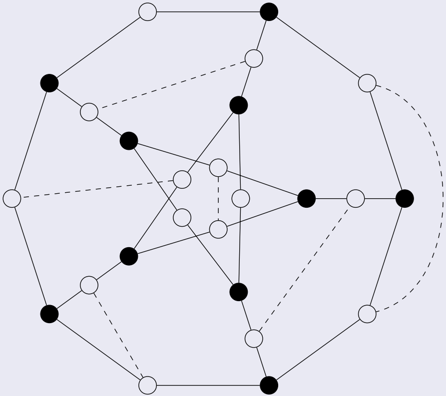

A general upper bound

Thm (Gravier, Moncel 07 [3])

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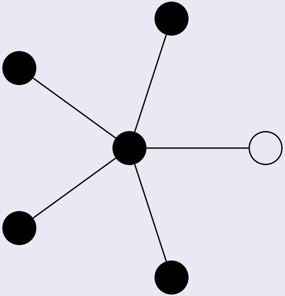
Thm (Gravier, Moncel 07 [3])

For all $n \geq 3$, there exist identifiable graphs with n vertices with $\gamma_{id}(G) = n - 1$.

Upper bound - example

Examples



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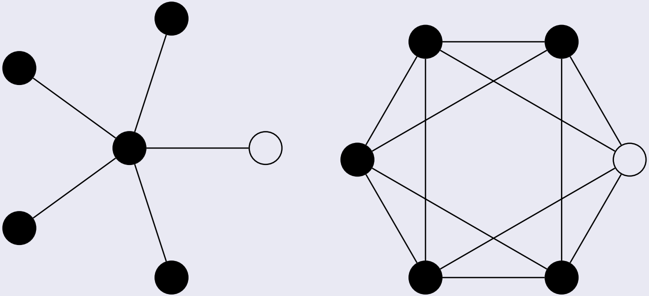

Upper bound and maximum degree

Remark

All these graphs have a high maximum degree $\Delta(G)$: $n-1$ or $n-2$.

Result - general case

Thm

Let G be a connected identifiable graph of maximum degree Δ.
Then $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^4)}$.
If G is regular, $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^2)}$.

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Sketch of the proof

- Greedily construct a 4-independant (resp. 2-independent) set S : distance between two vertices is at least 5 (resp. 3)
- take $C = V \backslash S$ as a code
- C must be modified locally

Connected cliques

- Take any Δ-regular graph H with m vertices
- replace any vertex of H by a clique of Δ vertices

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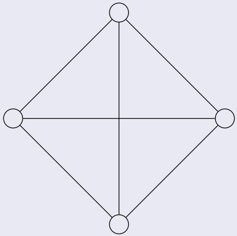
Example : $H = K_4$



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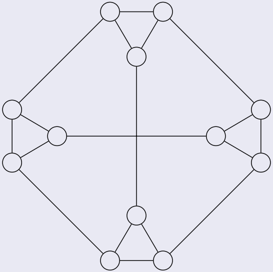

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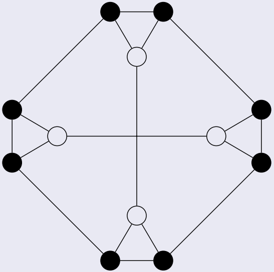

For every clique, at least $\Delta - 1$ vertices in the code
$\Rightarrow \gamma_{id}(G) \geq m \cdot (\Delta - 1) = n - \frac{n}{\Delta}$

Large codes in triangle-free graphs

Proposition

Let $K_{m,m}$ be the complete bipartite graph with $n = 2m$ vertices.
$id(K_{m,m}) = 2m - 2 = n - \frac{n}{\Delta}$.

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Thm (Bertrand et al. 05)

Let T_k^h be the k-ary tree with h levels and n vertices.

$$id(T_k^h) = \left\lceil \frac{k^2 n}{k^2 + k + 1} \right\rceil = n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}}.$$

Triangle-free graphs - Result

Thm

Let G be a connected triangle-free identifiable graph G with $n \geq 3$ vertices and maximum degree Δ.
Then $\gamma_{id}(G) \leq n - \frac{n}{3\Delta+3}$.
If G is regular, $\gamma_{id}(G) \leq n - \frac{n}{2\Delta+2}$.

Triangle-free graphs - Result

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Sketch of the proof

- Greedily construct an independent set S with special properties : $|S| \geq \frac{n}{\Delta+1}$
- Take $C = V \backslash S$ as a code
- Some vertices may not be identified correctly
- $\rightarrow$ locally modify C. It is possible to add not too much vertices to C

Graphs of girth at least 5

Thm

Let G be an identifiable graph with n vertices, of minimum degree $\delta \geq 2$ and girth $g \geq 5$.
Then $\gamma_{id}(G) \leq \frac{7n}{8} + 1$.

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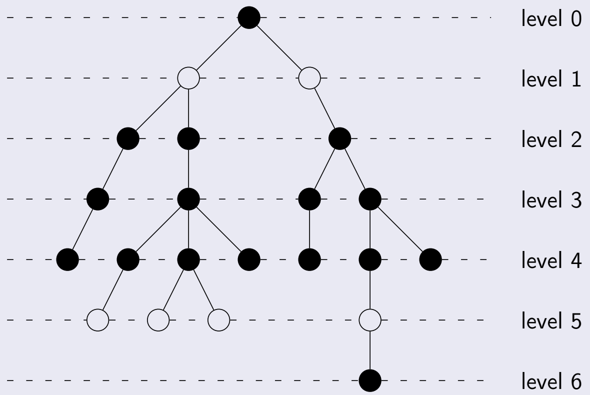
Sketch of the proof

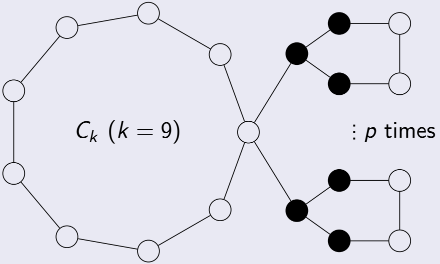
- Construct a DFS spanning tree T of G
- Partition the vertices into 4 classes V_0, V_1, V_2, V_3 depending on their level in T
- Take $C = V \setminus V_i$ as a code, $|V_i| \geq \frac{n}{4}$: $|V_i| \leq \frac{3n}{4}$
- C must be modified locally; the size of C might increase

Graphs of girth at least 5



Graphs of girth at least 5 - bad example

$G_{k,p} : \delta = 2, \Delta = p + 2, n = (5p + 1)k$



$\gamma_{id}(G_{k,p}) = 3pk = \frac{3}{5}(n - k) \rightarrow \frac{3n}{5}$

Summary

	arbitrary graphs	Δ-regular graphs
arbitrary graphs	$\left\langle n-\frac{n}{\Delta},\ n-\frac{n}{\Theta(\Delta^4)}\right\rangle$	$\left\langle n-\frac{n}{\Delta},\ n-\frac{n-1}{\Delta^2}\right\rangle$
triangle-free graphs	$\left\langle n-\frac{n}{\Delta-1+\frac{1}{\Delta}},\ n-\frac{n}{3\Delta+3}\right\rangle$	$\left\langle n-\frac{n}{\frac{2\Delta}{3}},\ n-\frac{n}{2\Delta+2}\right\rangle$

Summary

	arbitrary graphs	Δ-regular graphs
arbitrary graphs	$\left\langle n - \frac{n}{\Delta},\ n - \frac{n}{\Theta(\Delta^4)} \right\rangle$	$\left\langle n - \frac{n}{\Delta},\ n - \frac{n-1}{\Delta^2} \right\rangle$
triangle-free graphs	$\left\langle n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}},\ n - \frac{n}{3\Delta+3} \right\rangle$	$\left\langle n - \frac{n}{\frac{2\Delta}{3}},\ n - \frac{n}{2\Delta+2} \right\rangle$

	minimum degree $\delta \geq 2$
graphs of girth at least 5	$\left\langle \frac{3n}{5}, \frac{7n}{8} + 1 \right\rangle$

Upper bound : conjecture

Conjecture

Let G be a connected identifiable graph of maximum degree Δ.
Then $\gamma_{id}(G) \leq n - \frac{n}{\Delta}$.

Subcubic graphs

$\Delta = 3$

Let G be a subcubic identifiable graph with n vertices.

- $\gamma_{id}(G) \leq \frac{101n}{102}$
- If G is triangle-free, $\gamma_{id}(G) \leq \frac{11n}{12}$
- If G is cubic, $\gamma_{id}(G) \leq \frac{8n}{9}$
- If G is cubic and triangle-free, $\gamma_{id}(G) \leq \frac{7n}{8}$

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- If G is cubic and triangle-free, $\gamma_{id}(G) \leq \frac{7n}{8}$

Recall:

There are infinitely many cubic graphs G such that $\gamma_{id}(G) = \frac{2n}{3}$.

Subcubic graphs

Thm

Let G be a Hamiltonian subcubic graph with $n \geq 4$ vertices.
Then $\gamma_{id}(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor$.

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Sketch of the proof

- Consider a Hamiltonian path P
- Start with $C = V$
- Remove 1 from 4 vertices on P

Corollary for cubic graphs

Lemma (Kawarabayashi et al., 2002)

Every 2-connected cubic graph has a 2-factor in which every component is a cycle of length at least 4.

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Every 2-connected cubic graph has a 2-factor in which every component is a cycle of length at least 4.

Corollary

Let G be a 2-connected cubic graph with n vertices.
Then $\gamma_{id}(G) \leq \frac{3n}{4}$.

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