Bounding the identifying code number of a graph using its degree parameters

(a probabilistic approach)

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GT G&A - 29th April 2011

joint work with **Guillem Perarnau** (UPC, Barcelona)

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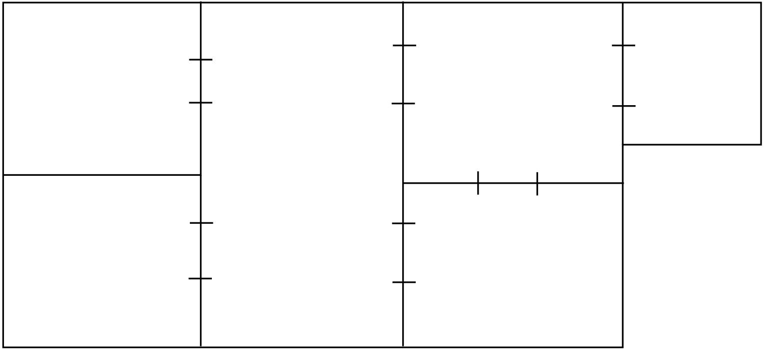
Locating a burglar in a museum



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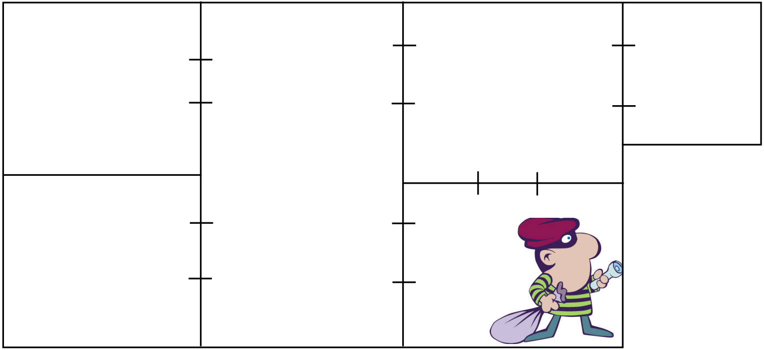

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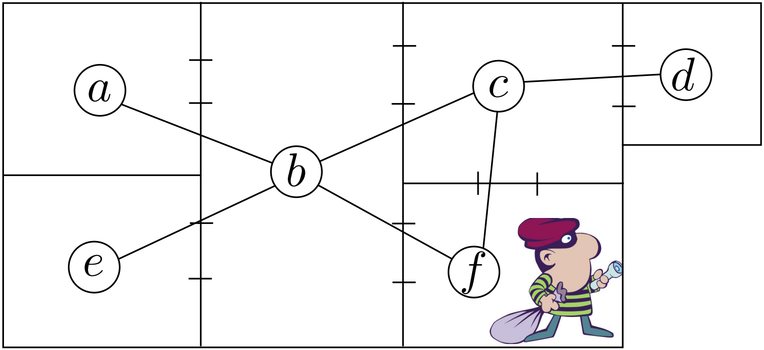

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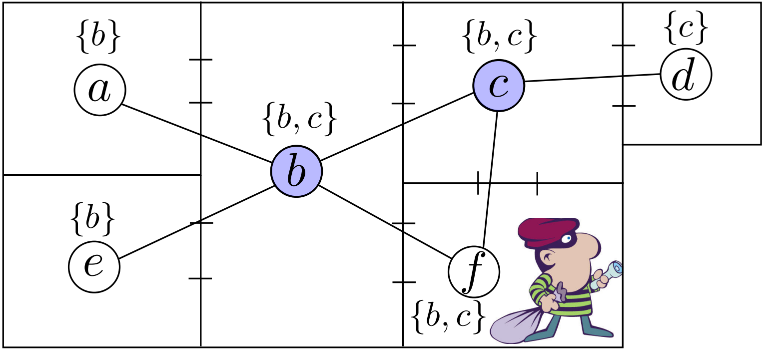

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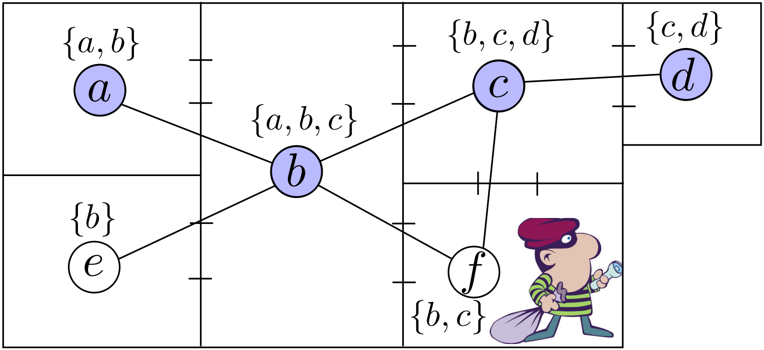

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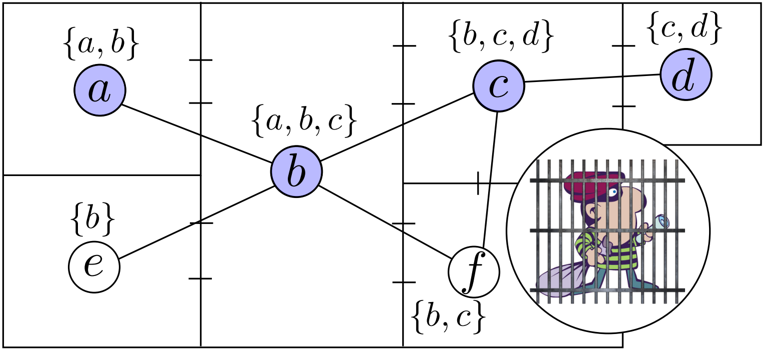

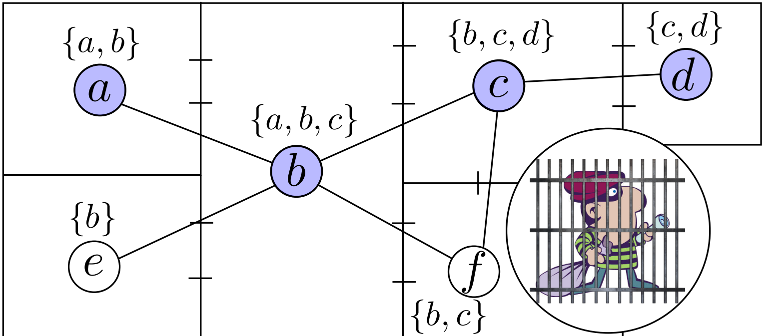

How many detectors do we need?

Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a dominating set in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating code in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

Let $N[u]$ be the set of vertices v s.t. $d(u,v) \leq 1$

Remark

Not all graphs have an identifying code!

Twins $=$ pair u, v such that $N[u] = N[v]$.

A graph is identifiable iff it is twin-free (i.e. it has no twins).

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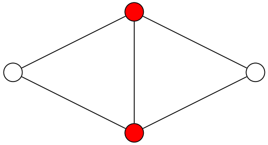

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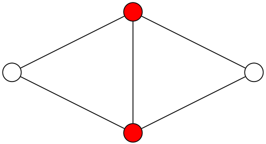

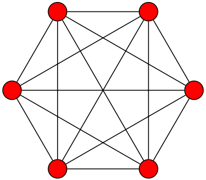

Previous results (1/2)

Theorem (Karpovsky, Chakrabarty, Levitin, 1998 + Gravier, Moncel, 2007)

Let G be an identifiable graph with at least one edge, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

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Let G be an identifiable graph with maximum degree d, then

$$\tfrac{2n}{d+2} \leq \gamma^{\text{ID}}(G)$$

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Conjecture (F., Klasing, Kosowski, Raspaud, 2009+)

Let G be a connected nontrivial identifiable graph of max. degree d. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1)$$

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This would be tight. True for $d = 2$ and $d = n - 1$.

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Theorem (F., Guerrini, Kovse, Naserasr, Parreau, Valicov, 2011)

Let G be a connected identifiable graph of maximum degree d. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^5)}$$

If G is d-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$

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Theorem (F., Klasing, Kosowski, Raspaud, 2009+)

Let G be a connected identifiable triangle-free graph of max. degree d. Then

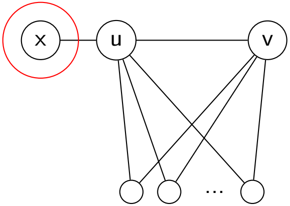
$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{3d+3}$$

If G is d-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{2d+2}$

Forced vertices

u, v such that $N[v] \Delta N[u] = \{x\}$

Then $x \in C$, forced by uv.



Proposition

Let $f(G)$ be the proportion of non forced vertices of G. Then

$$\frac{1}{d+1} \leq f(G) \leq 1$$

This result is tight for a graph of max. degree $d = n - 1$.
Actually, we believe $\frac{1}{d} - O(\frac{1}{n}) \leq f(G)$ holds.

Sketch of Proposition's proof

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Lemma Bertrand, Hudry, 2005

Let G be an identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex y in $N[x]$.

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Corollary

The set S of non-forced vertices forms a dominating set. Hence $|S| \geq \frac{n}{d+1}$.

Upper bounds for $\gamma^{\text{ID}}(G)$

Theorem (F., Perarnau, 2011+)

Let G be an identifiable graph of maximum degree d having no isolated vertices. Then

$$\gamma^{\text{ID}}(G) \leq \begin{cases} \left(1 - \frac{f(G)}{\Theta(d^{3/2})}\right) n & \text{if } f(G) = \omega(d^{-1/2}) \\ \left(1 - \frac{f(G)^2}{\Theta(d)}\right) n & \text{if } f(G) = O(d^{-1/2}) \end{cases}$$

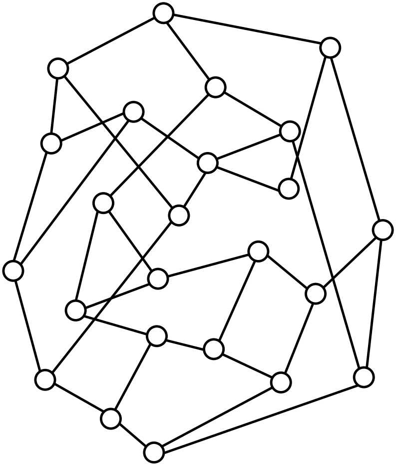
Corollary

By the previous Proposition we know $f(G) \geq \frac{1}{d+1}$, then

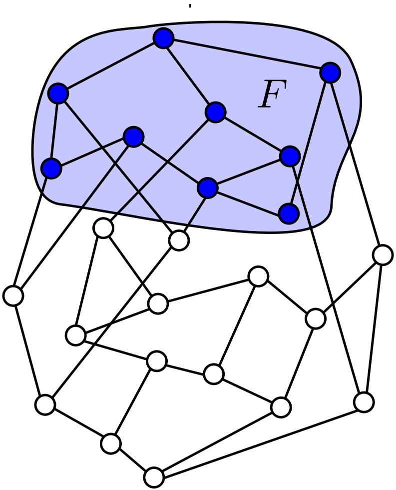
$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$$

Moreover if G is d-regular, $f(G) = 1$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^{3/2})}$.

Proof - select a set at random...

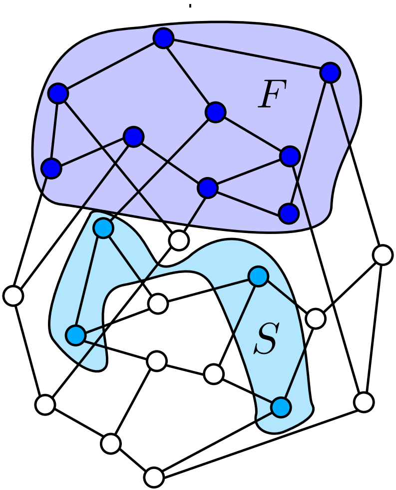


- F: forced verticed



- F: forced verticed

- Select a random set S from $V' = V \setminus F$: each vertex $v \in S$ with prob. p.



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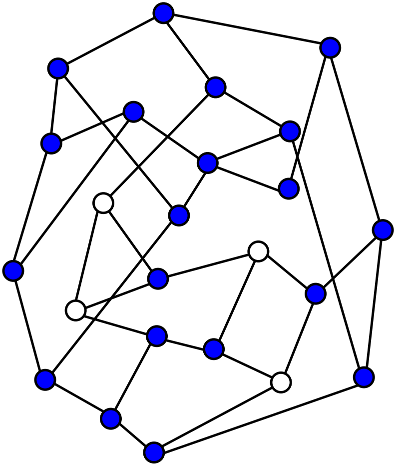
- $\mathcal{C} = V \setminus S$



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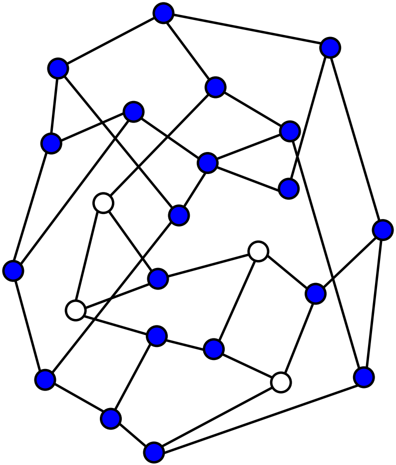

Proof - Lovász Local Lemma

Theorem (Weighted Local Lemma: particular case of the Local Lemma Erdős, Lovász, 1973 - Molloy, Reed, 2001[1])

Let $0 < p \leq \frac{1}{4}$ and $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a set of "bad" events such that each E_i is mutually independent of $\mathcal{E} \setminus (\mathcal{D}_i \cup \{E_i\})$ where $\mathcal{D}_i \subseteq \mathcal{E}$, and

- $Pr(E_i) \leq p^{t_i}$
- $\sum_{E_j \in D_i} (2p)^{t_j} \leq \frac{t_i}{2}$

Then $Pr(\bigcap_{i=1}^{M} \overline{E_i}) \geq \prod_{i=1}^{M} (1 - (2p)^{t_i}) \geq \exp\left\{ -\sum_{i=1}^{m} (2p)^{t_i} \right\} > 0.$

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$\Longrightarrow$ If the dependencies are "rare":

with non-zero probability none of the bad events occur

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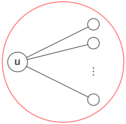
Set the bad events...



$p^{d+1} \leq Pr(A_u) \leq p^2$

Event A_u

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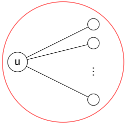

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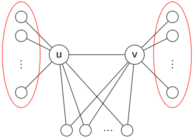

Event $B_{u,v}$

$$p^{2d-2} \leq Pr(B_{u,v}) \leq p^2$$

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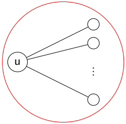

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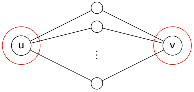
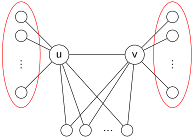


Event $C_{u,v}$

$Pr(C_{u,v}) = p^2$



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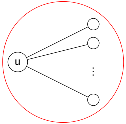

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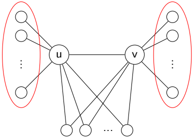

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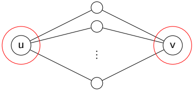

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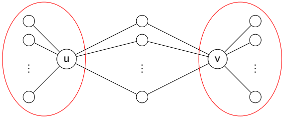

Event $D_{u,v}$

$$p^{2d} \leq Pr(D_{u,v}) \leq p^4$$

Proof (regular case) - Lovász Local Lemma, technicalities

Theorem (Weighted Local Lemma)

Bad events: $\mathcal{E} = \{E_1, \ldots, E_m\}$.

- $Pr(E_i) \leq p^{t_i}$
- $\sum_{E_j \in D_i} (2p)^{t_j} \leq \frac{t_i}{2}$

Then $Pr(\bigcap_{i=1}^{M} \overline{E_i}) \geq \prod_{i=1}^{M} (1 - (2p)^{t_i}) > 0.$

Build the event-intersection table:

	A	B	C	D
A	d^2	$d^3 - 2d^2 + 2d$	$d^4 + d^2 - 2d$	$d^2 - d + 1$
B	$2(d^2 - d + 1)$	$2d^3 - 4d^2 + 4d - 1$	$2d(d^3 - 3d^2 + 4d - 2)$	$d^2 - d$
C	$2d^2 + 2$	$2d(d^2 - 2d + 2)$	$2d^4 - 6d^3 + 9d^2 - 5d - 1$	$d^2 + d - 2$
D	$d + 2$	$d^2 + d$	$d^3 - d$	$2d - 4$

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$$d^2(2p)^{d+1} + (d^3 - 2d^2 + 2d)(2p)^2 + (d^4 + d^2 - 2d)(2p)^3 + (d^2 - d + 1)(2p)^2 \leq \frac{2}{2} = 1$$

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Taking $p = \frac{1}{kd^{3/2}} \implies$ LLL can be applied

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Bu by the LLL we know more:

$$\Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > \exp\left\{-\sum_{i=1}^{m} (2p)^{t_i}\right\}$$

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Bu by the LLL we know more:

$$\Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > \exp\left\{-\frac{5}{k^2 d^2} n\right\}$$

The probability to have a good set S is at least $\exp\left\{-\frac{5}{k^2 d^2} n\right\}$

Proof (regular case) - concentration inequality

Theorem (Chernoff bound)

Let $X_1, \ldots, X_n$ a set of i.i.d random variables s.t. $Pr(X_i = 1) = p$ and $Pr(X_i = 0) = 1 - p$ and $X = \sum X_i$. Then

$$\Pr(\mathbb{E}(X) - X > \alpha) \leq \exp\left\{-\frac{\alpha^2}{2np}\right\}$$

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Define the random variables

$$X_i = \begin{cases} 1 & \text{if } v_i \in C \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\Pr\left(\mathbb{E}(X) - X > \frac{1}{cd^{7/4}} n\right) \leq \exp\left\{\frac{k}{2c^2 d^2} n\right\}$$

Proof (regular case) - concentration inequality

Theorem (Chernoff bound)

Let $X_1, \ldots, X_n$ a set of i.i.d random variables s.t. $Pr(X_i = 1) = p$ and $Pr(X_i = 0) = 1 - p$ and $X = \sum X_i$. Then

$$\Pr(\mathbb{E}(X) - X > \alpha) \leq \exp\left\{-\frac{\alpha^2}{2np}\right\}$$

Define the random variables

$$X_i = \begin{cases} 1 & \text{if } v_i \in C \\ 0 & \text{otherwise} \end{cases}$$

Then,

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Probability that S is too small: at most $\exp\left\{-\frac{k}{2c^2d^2}n\right\}$

Proof (regular case) - size of the code

$$\Pr(S \text{ good}) - \Pr(S \text{ too small}) > 0$$

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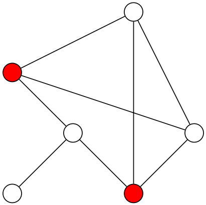
$$|\mathcal{C}| = |V \setminus S| \leq n - \frac{1}{2.52d^{3/2}}n$$

Weak twins = pair u, v such that $N(u) = N(v)$ but $u \not\sim v$.



Graphs without weak twins

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Theorem (F., Perarnau, 2011+)

Let G be a d-regular identifiable graph without weak twins, then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{41d}$$

Moreover there are graphs such that $\gamma^{\text{ID}}(G) = n - \frac{n}{d}$, so it is (asymptotically) best possible.

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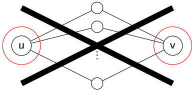

Event $C_{u,v}$

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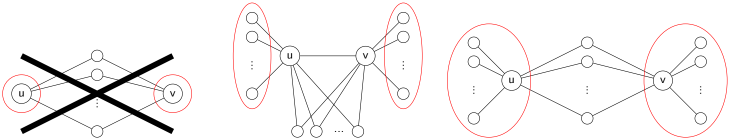
Proof: Delete events C, and split B and D depending on their size.



Event $C_{u,v}$ Event $B_{u,v}$ Event $D_{u,v}$

Let G be a d-regular graph.

$$\gamma^{\text{ID}}(G) \geq \frac{2n}{d+2}$$ Karpovsky et al. (1998)

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1)$$ Conjecture - F. et al. (2009+)

Where are most of the d-regular graphs?

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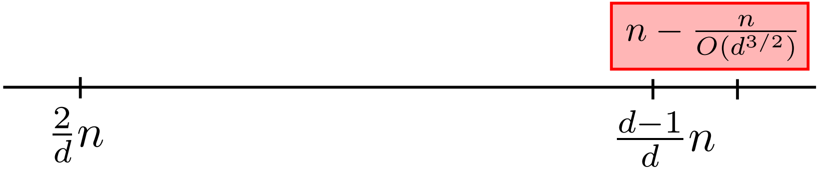

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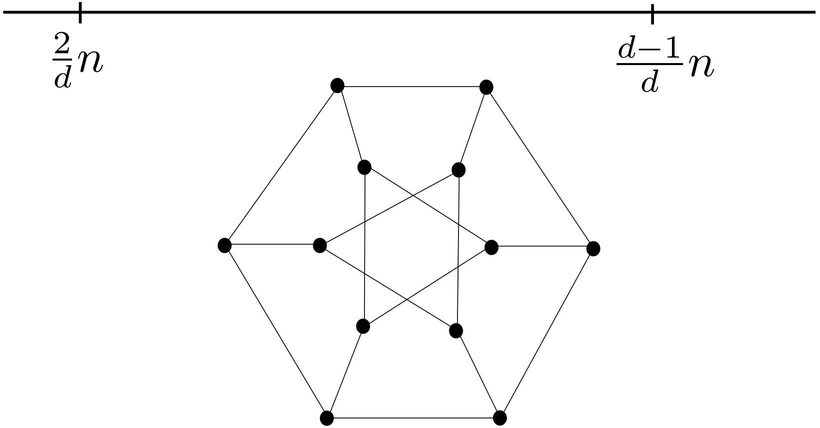

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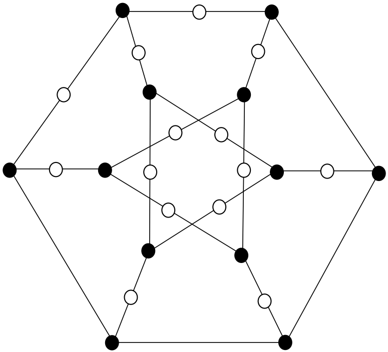
$\frac{2}{d}n$ $\frac{d-1}{d}n$



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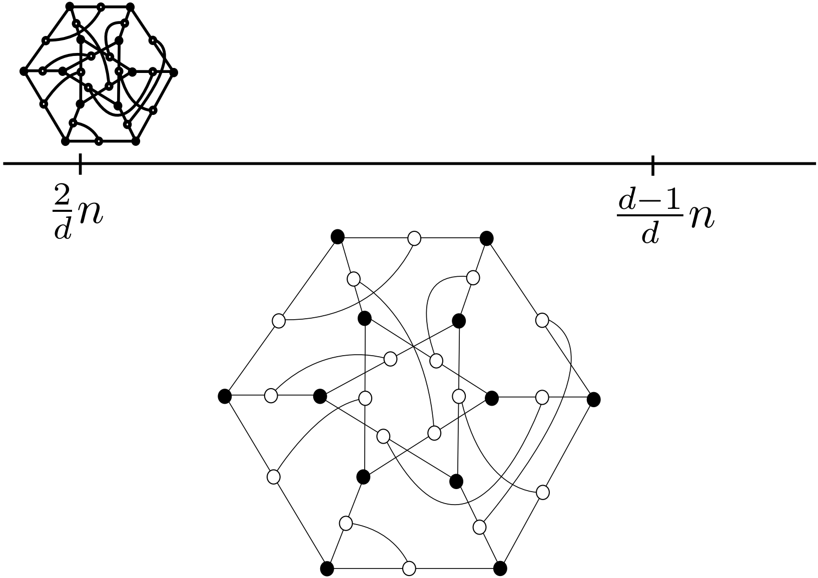

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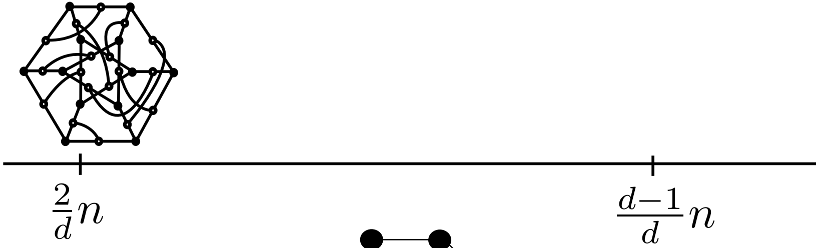

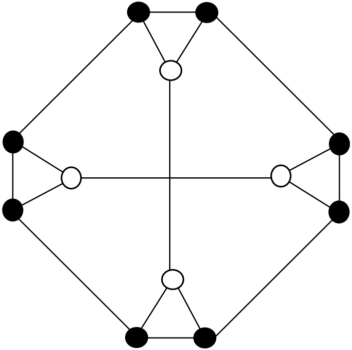

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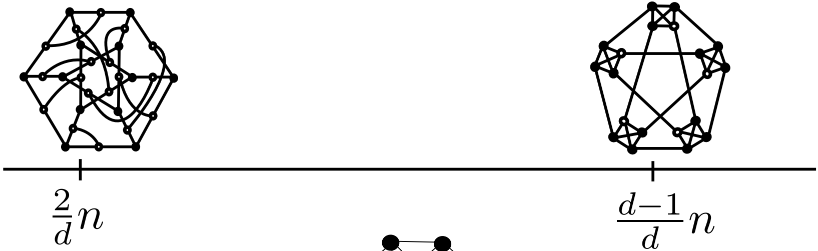

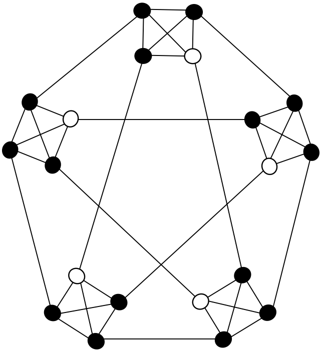

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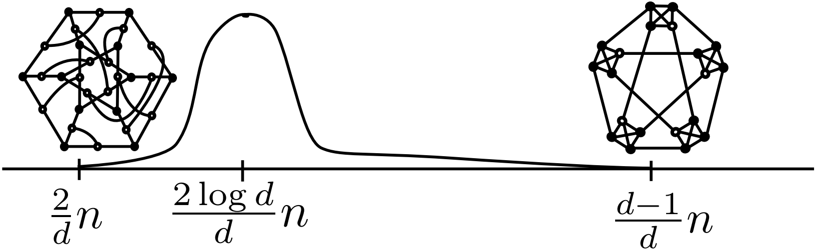

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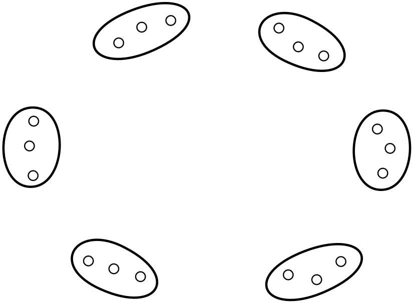
Let G be a random d-regular graph. Then a.a.s.

$$(1+o_d(1))\frac{\log d}{d}n \leq \gamma^{\text{ID}}(G) \leq (1+o_d(1))\frac{2\log d}{d}n$$

The pairing model (a.k.a. configuration model) - Bollobás, 1980

Probability space $\mathcal{G}^*_{n,d}$ of d-regular multigraphs on n vertices.



- Take nd vertices grouped in n buckets of size d

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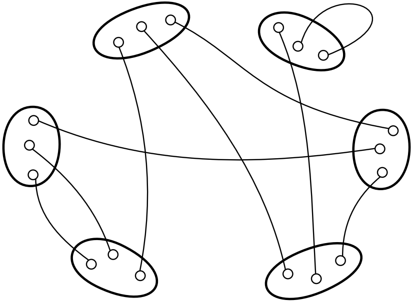

- Take nd vertices grouped in n buckets of size d
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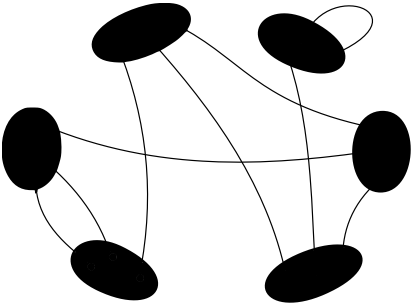

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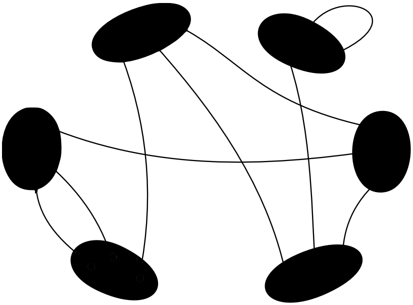

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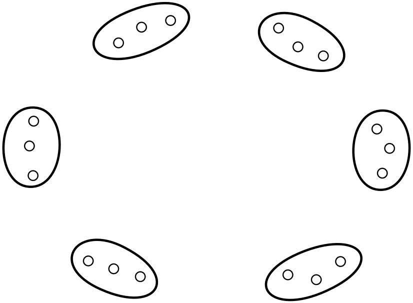

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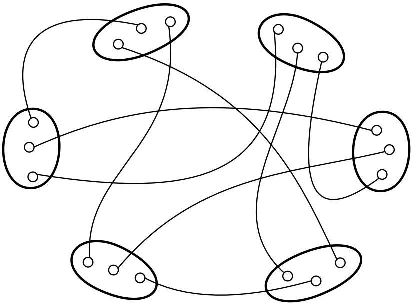

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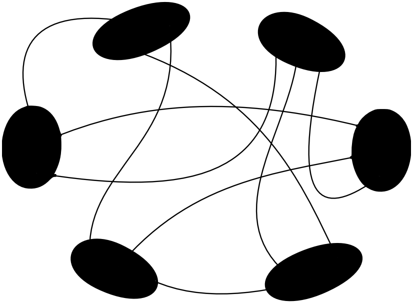

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Let $G \in \mathcal{G}^*_{n,d}$. Then $Pr(G \text{ is simple}) \longrightarrow e^{\frac{1-d^2}{4}} > 0$

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Graphs with girth at least 5

Proposition (F., Perarnau, 2011+)

Let G be a d-regular graph with girth at least 5. Then

$$\gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d}n$$

2-dominating is "almost sufficient" to identify.

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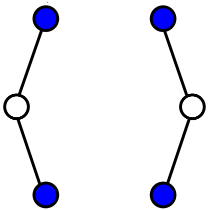

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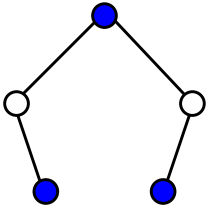

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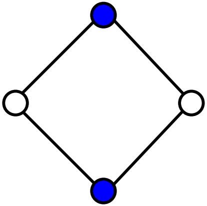

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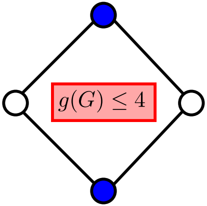
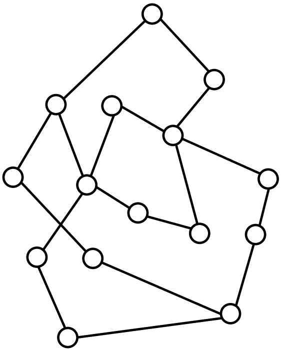


$g(G) \geq 5$ makes identifying easier.

Sketch of the proof: construct 2-dominating set D

- $S \subseteq V$ at random, each element with probability p.



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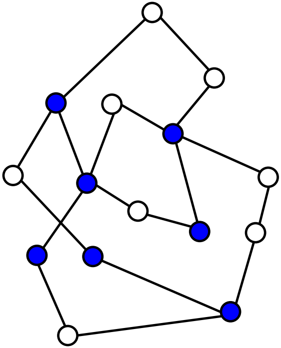

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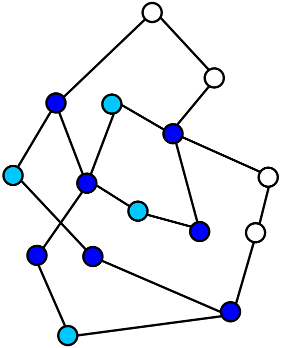

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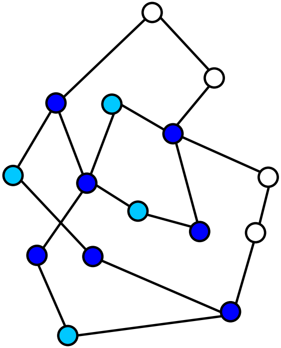
$$X_v = \begin{cases} 0 & \text{if } |N[v] \cap S| \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

$$Pr(X_v = 1) = (1-p)^{d+1} + (d+1)p(1-p)^d$$



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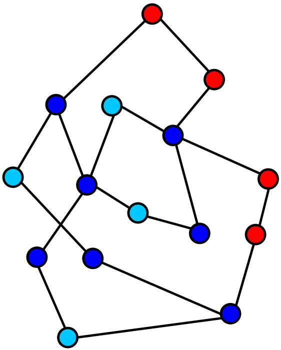

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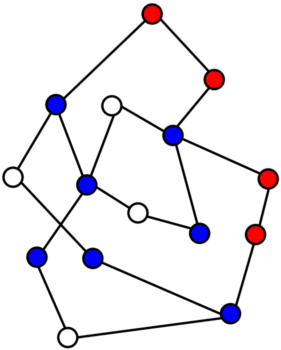
- $X(S) = \sum X_v$ (# non 2-dominated).
- $\mathcal{C} = S \cup \{v : X_v = 1\}$, $p = \frac{\log d}{d}$

$$\mathbb{E}(|D|) = \mathbb{E}(|S|) + X(S) \leq \frac{2 \log d}{d} n$$



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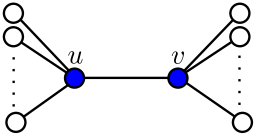
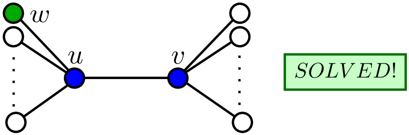


PROBLEM!

Sketch of the proof: identifying code



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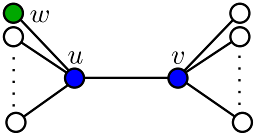
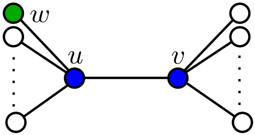


SOLVED!

$$Y_{uv} = \begin{cases} 1 & \text{if } \end{cases}$$ 
$$\qquad\qquad 0 \quad \text{otherwise}$$

$$Pr(Y_{uv} = 1) = p^2(1-p)^{2d-2}$$ **SMALL**

Sketch of the proof: identifying code



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$$\mathcal{C} = S \cup \{v : X_v = 1\} \cup \{w : w \in N(u),\ Y_{uv} = 1\},\ p = \tfrac{\log d}{d}$$

$$\mathbb{E}(|\mathcal{C}|) = (1 + o_d(1))\frac{2\log d}{d}n$$

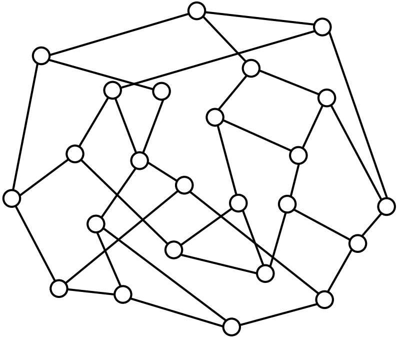
Back to random regular graphs - upper bound

Theorem (F., Perarnau, 2011+)

Let G be a random d-regular graph. Then a.a.s.

$$\gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d}n$$

Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n, d)$



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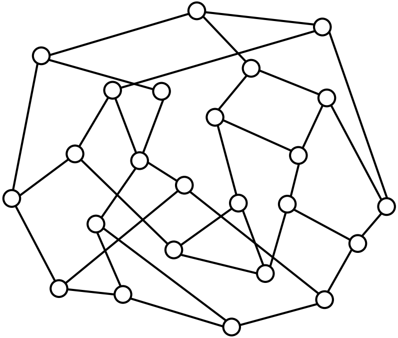
$$\Pr(G \text{ identifiable}) \xrightarrow{n} 1$$



Theorem (F., Perarnau, 2011+)

Let G be a random d-regular graph. Then a.a.s.

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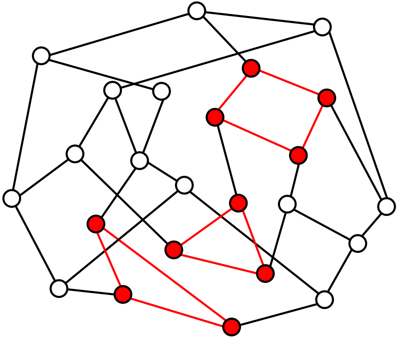
$\mathbb{E}(C_3\text{'s}) = e^{\frac{(d-1)^3}{6}}$ $\mathbb{E}(C_4\text{'s}) = e^{\frac{(d-1)^4}{8}}$



Theorem (F., Perarnau, 2011+)

Let G be a random d-regular graph. Then a.a.s.

$$\gamma^{\text{ID}}(G) \le (1 + o_d(1))\frac{2\log d}{d}n$$

Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n, d)$

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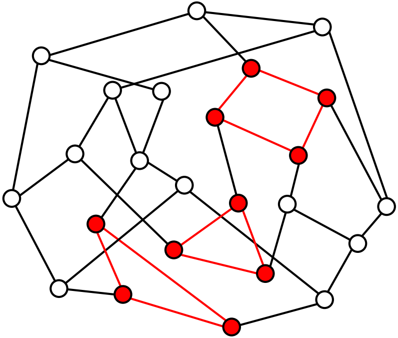
$$\Pr(\#C_3 > \log\log n) \longrightarrow 0$$

$$\Pr(\#C_4 > \log\log n) \longrightarrow 0$$



Back to random regular graphs - upper bound

Theorem (F., Perarnau, 2011+)

Let G be a random d-regular graph. Then a.a.s.

$$\gamma^{\text{ID}}(G) \leq (1 + o_d(1))\frac{2\log d}{d}n$$

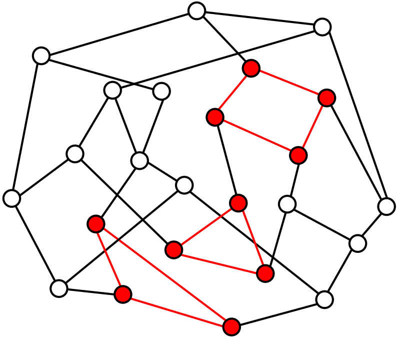
Let G be a d-regular graph of order n, taken u.a.r.: $G \in \mathcal{G}(n,d)$

$$\Pr(G \text{ identifiable}) \xrightarrow{n} 1$$

$$\mathbb{E}(C_3\text{'s}) = e^{\frac{(d-1)^3}{6}} \qquad \mathbb{E}(C_4\text{'s}) = e^{\frac{(d-1)^4}{8}}$$

$$\Pr(\#C_3 > \log\log n) \longrightarrow 0$$

$$\Pr(\#C_4 > \log\log n) \longrightarrow 0$$



$$\gamma^{\text{ID}}(G) \leq |\mathcal{C}| = (1 + o_d(1))\frac{2\log d}{d}n \textbf{ a.a.s.}$$

Summary for d-regular graphs

	Identifying codes	Dominating sets
girth 3	$n - \frac{n}{\Theta(d^{3/2})}$ [FP] (Conj.: $n - \frac{n}{\Theta(d)}$ [FKKR])	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
girth 3 and weak-twin-free	$n - \frac{n}{\Theta(d)}$ [FP], [FKKR]	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
girth 4	$n - \frac{n}{\Theta(d)}$ [FKKR], [FKKR]	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
girth 5	$\Theta\left(\frac{\log d}{d} n\right)$ [FP], [1]	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
random d-regular graphs (a.a.s.)	$\Theta\left(\frac{\log d}{d} n\right)$ [FP], [1]	$\Theta\left(\frac{\log d}{d} n\right)$ [FP], [1]

FP: F., Perarnau, 2011+
FKKR: F., Klasing, Kosowski, Raspaud, 2009+
AS: Alon and Spencer, *The probabilistic method*, 2000
TY: Thomassé and Yeo, 2007
1: relatively easy LB for dominating sets in RRG, no reference found

THANKS FOR YOUR ATTENTION!

	Identifying codes	Dominating sets
girth 3	$n - \frac{n}{\Theta(d^{3/2})}$ [FP] (Conj.: $n - \frac{n}{\Theta(d)}$ [FKKR])	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
girth 3 and weak-twin-free	$n - \frac{n}{\Theta(d)}$ [FP], [FKKR]	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
girth 4	$n - \frac{n}{\Theta(d)}$ [FKKR], [FKKR]	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
girth 5	$\Theta\left(\frac{\log d}{d} n\right)$ [FP], [1]	$\Theta\left(\frac{\log d}{d} n\right)$ [AS], [TY]
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