Random subgraphs make identification affordable

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joint work with Guillem Perarnau *and* Oriol Serra.

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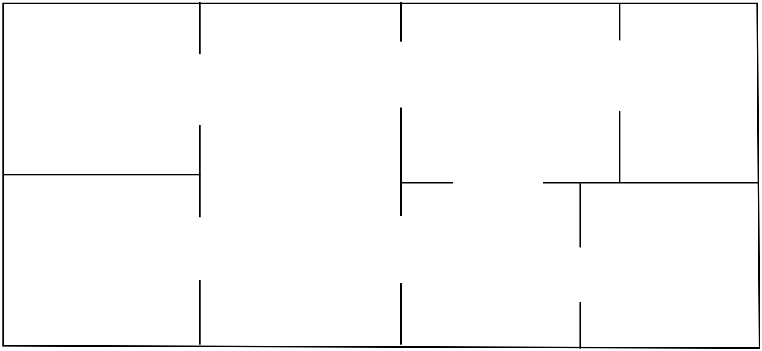
Locating a burglar in museum



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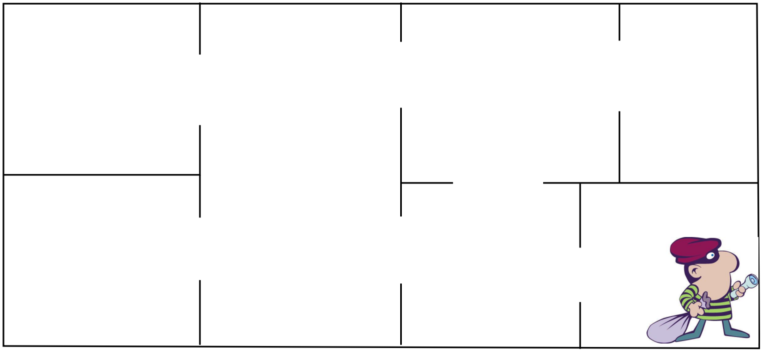

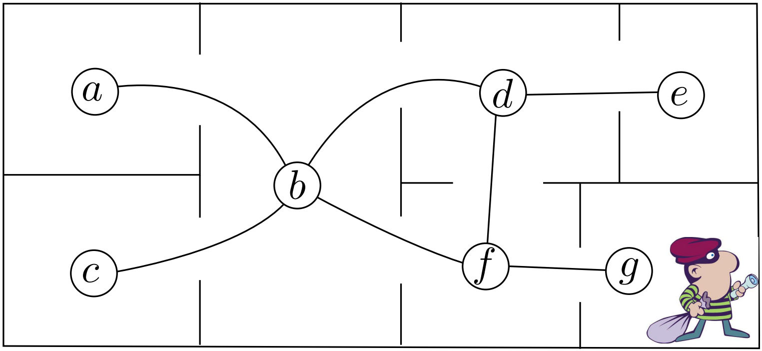
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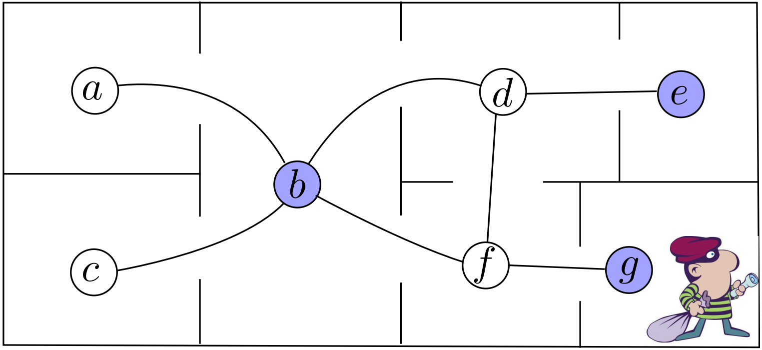

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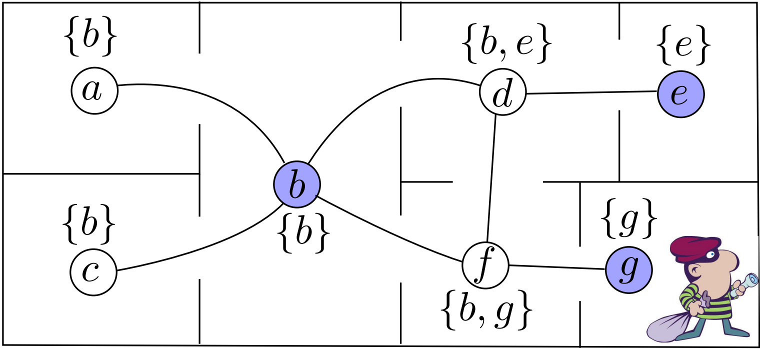

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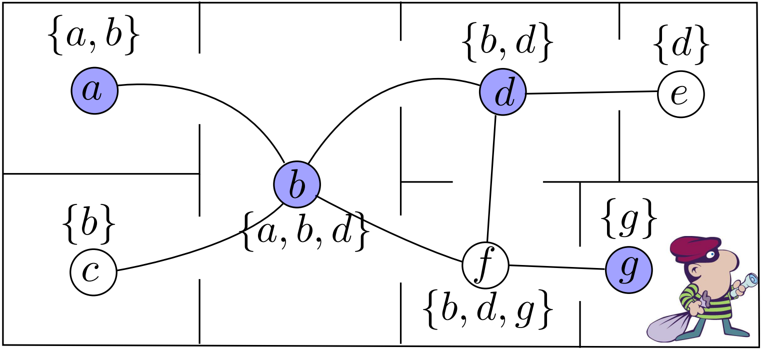
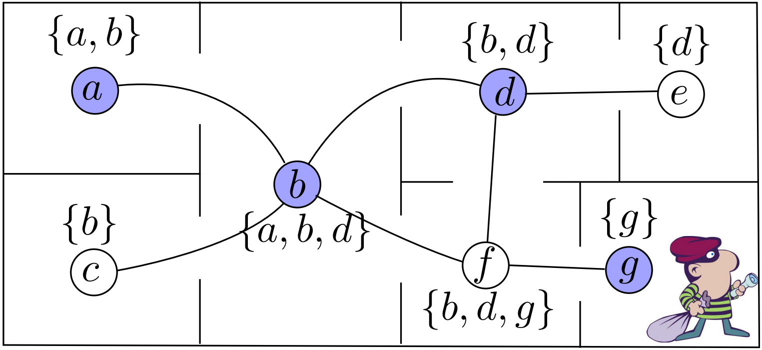
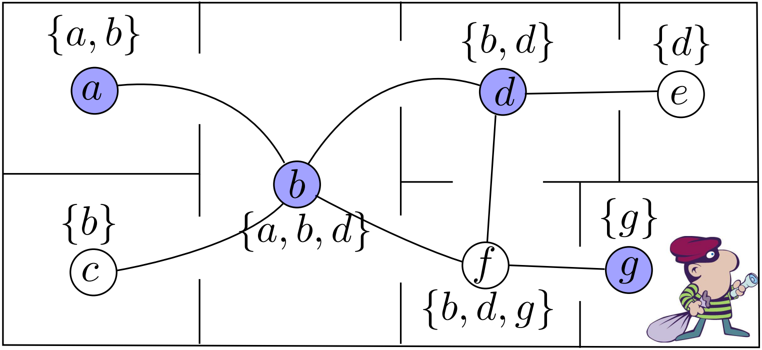




$N[v] = N(v) \cup \{v\}$

$\mathcal{C}$ is an identifying code of G:

- for every $u \in V$, $N[v] \cap \mathcal{C} \neq \emptyset$ (domination).
- $\forall u \neq v$ of V, $N[u] \cap \mathcal{C} \neq N[v] \cap \mathcal{C}$ (separation).



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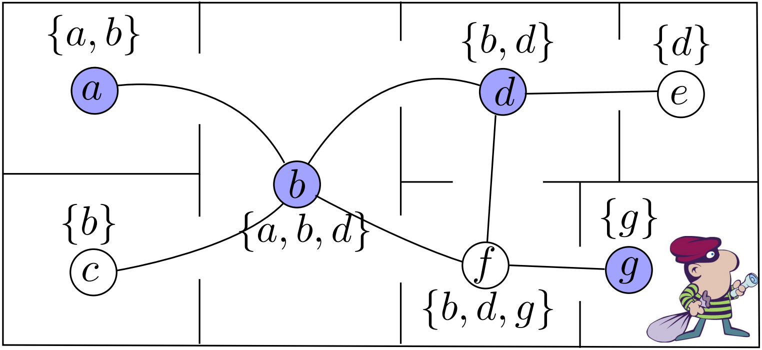
$\gamma^{\text{ID}}(G)$: identifying code number , minimum size of an identifying code of G.



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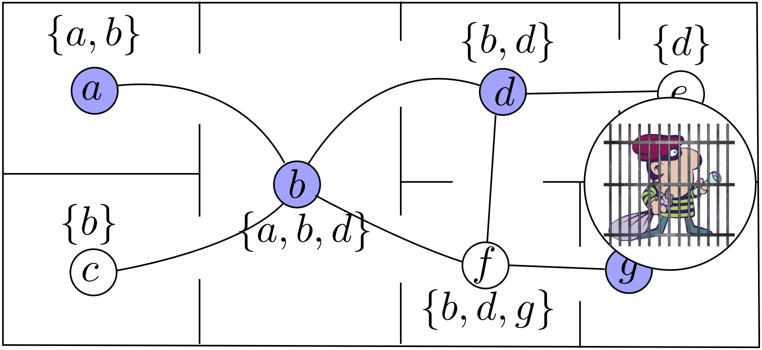

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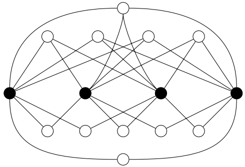

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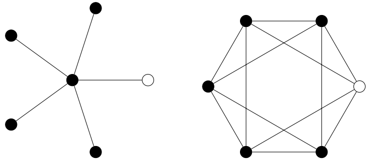

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Let G be a nonempty graph on n vertices, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

There exist arb. large connected r–regular graphs G_r with

$$\gamma(G_r) = \frac{1}{r} n \quad \text{and} \quad \gamma^{\text{ID}}(G_r) = \left(1 - \frac{1}{r}\right) n\ .$$

Question: Is every graph *close* to admit a small identifying code?

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Can we delete (or add) a small number of edges such that the remaining graph has an optimal identifying code?

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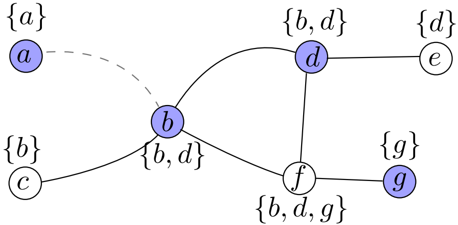

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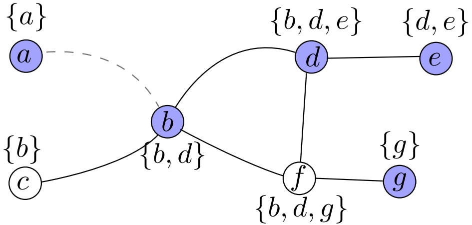

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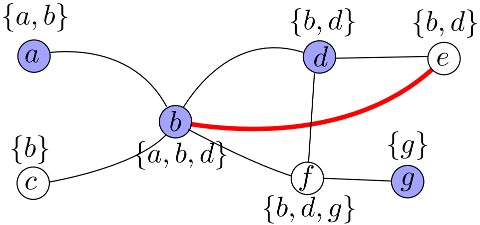

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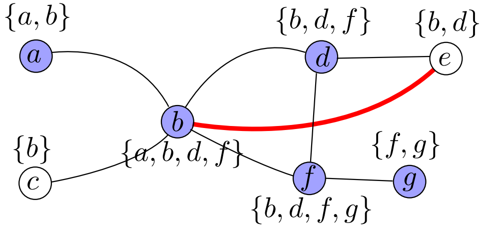

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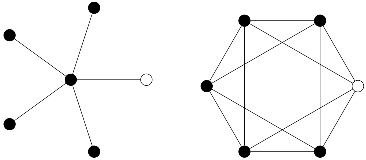

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Observation: for any $H \subseteq G$,

$$\gamma^{\text{ID}}(H) \geq \gamma(H) \geq \gamma(G)\,.$$

Question

Does G admit a spanning subgraph H satisfying

$$\gamma^{\text{ID}}(H) = \Theta\left(\gamma(G)\right)\,?$$

Random structures behave nicely

For any $0 < p < 1$, let $b = 1/(1-p)$. Then, whp

$$\gamma(G(n,p)) = (1 + o(1))\frac{\log n}{\log b} .$$

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For any $0 < p < 1$, let $q = p^2 + (1-p)^2$. Then, whp

$$\gamma^{\text{ID}}(G(n,p)) = (1+o(1))\frac{2\log n}{\log(1/q)} = \Theta(\gamma(G))\ .$$

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Intuition:

If G has a random structure, then domination is almost enough to identify it.

Idea:

Introduce randomness in G by removing edges to decrease $\gamma^{ID}(G)$.

Every graph is close to admit a small identifying code

Theorem (F., Perarnau, Serra 2013+)

For any graph G on n vertices with maximum degree $\Delta = \omega(1)$ and minimum degree $d \geq 66 \log \Delta$, there exists a subset of edges $F \subset E(G)$ of size

$$|F| = O(n \log \Delta) \ ,$$

such that

$$\gamma^{\text{ID}}(G \setminus F) = O\left(\frac{n \log \Delta}{d}\right) \ .$$

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If Δ is bounded, for any $H \subseteq G$,

$$\gamma^{\text{ID}}(H) \geq \gamma(G) \geq \frac{n}{\Delta + 1} = \Omega(n) .$$

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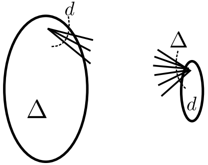
Consider $K_{d,\Delta}$, with $d = \frac{1}{2} \log_2 \Delta$ $(\Delta = 4^d)$.

Let $H \subseteq K_{d,\Delta}$ and let $\mathcal{C}$ be a code of H. For $v \in V$ there are at most 2^d candidates for $N_H(v) \cap \mathcal{C}$.



Then

$$\gamma^{\text{ID}}(H) = (1 - o(1))n \ ,$$

for any $H \subseteq K_{d,\Delta}$.

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Consider G to be a random r-regular graph. With high probability

$$\gamma(G) = (1 + o(1))\frac{n \log r}{r} \ ,$$

thus, for any $H \subseteq G$

$$\gamma^{\text{ID}}(H) \geq \gamma(G) \geq \frac{n \log r}{r} \ .$$

It is not clear whether $\log \Delta$ can be replaced by $\log d$.

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Proposition

For any set $F \subseteq E(K_n)$ of size $o(n \log n)$ we have $\gamma^{\text{ID}}(K_n \setminus F) = \omega\left(\log n\right)$.

Let G_r denote the disjoint union of cliques of size $r+1$, then for any set $F \subseteq E(G_r)$ of size $o(n \log r)$,

$$\gamma^{\text{ID}}(G_r \setminus F) = \omega\left(\frac{n \log r}{r}\right) \ .$$

Again, not clear whether $\log \Delta$ can be replaced by $\log d$.

Select $\mathcal{C}$ at random: each vertex with probability

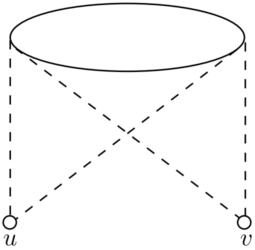
$$p = O\left(\frac{\log \Delta}{d}\right) .$$

For any pair of vertices $u, v \in V$ we want

$$N[u] \cap \mathcal{C} \neq N[v] \cap \mathcal{C} .$$

In the worst case

$$N[u] = N[v] .$$



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Goal: Remove edges to make

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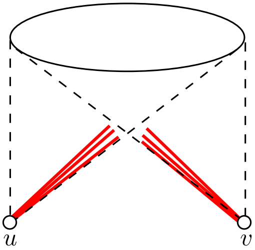
large enough $\Rightarrow$ $\mathcal{C}$ intersects it *whp*.



Random sets as codes

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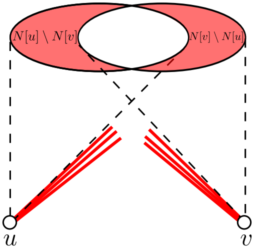

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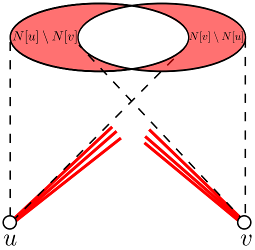


It suffices to remove $\Theta(\log \Delta)$ edges!

Given $\mathcal{C} \subseteq V$ we remove from G the edge uv with probability

$$p_{u,v} = \Theta\left(\frac{\log \Delta}{d_{\mathcal{C}}(u)} + \frac{\log \Delta}{d_{\mathcal{C}}(v)}\right) ,$$

if it is incident to $\mathcal{C}$.

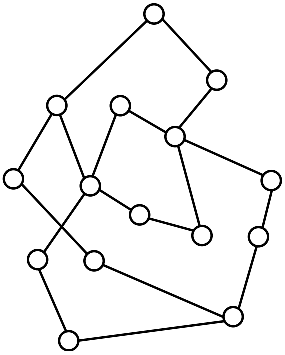
For any $u \in V$, we expect to remove

$$\Theta(\log \Delta) ,$$

incident edges.

The expected number of removed edges is

$$O(n \log \Delta) .$$



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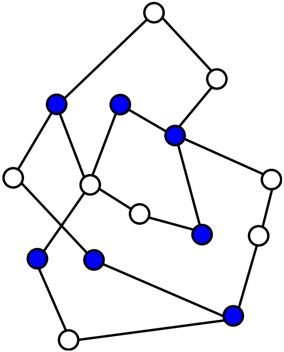

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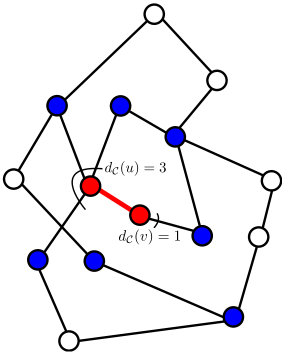

Candidate subgraph $G(\mathcal{C})$

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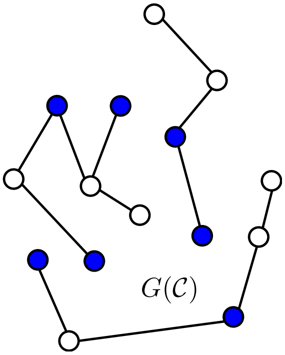

Proof in 4 steps

1.- Select $\mathcal{C}$ at random. Then,

$$|\mathcal{C}| > 2\frac{n \log \Delta}{d} ,$$

with exponentially small probability.

2.- Use Lovász Local Lemma to show that a random set $\mathcal{C}$ and the random subgraph $G(\mathcal{C})$ satisfy

i) $d_{\mathcal{C}}(v)$ are concentrated around $d(v)p \; \forall v \in V$.
ii) $N_{G(\mathcal{C})}[u] \cap \mathcal{C} \neq N_{G(\mathcal{C})}[v] \cap \mathcal{C} \; \forall u, v \in V$ at distance at most 2 (local separation).
with exponentially large probability.

3.- Add a dominating set to $\mathcal{C}$ which has size at most

$$\frac{n \log d}{d} ,$$

to take care of global separation property.

4.- The conditioned expected number of deleted edges is

$$O(n \log \Delta) .$$

Adding edges

Adding:

Since,

$$\gamma^{\text{ID}}(\overline{G}) = \Theta(\gamma^{\text{ID}}(G))$$

an analogous result holds for adding edges instead of removing them.

Deleting + Adding:

Question

Can we improve the previous result if we are allowed to delete and add edges?

Of course! : Remove all the edges of G and add the edges to construct an optimal graph.

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Deleting + Adding:

Question

Can we improve the previous result if we are allowed to delete and add a small amount of edges?

NO : If we delete/add at most $O(n \log \Delta)$ edges, the previous result cannot be improved.

Open Questions I

If $\Delta = Poly(d)$, then the theorem is asymptotically tight ($\log \Delta = \Theta(\log d)$).

Due to domination property, for some graphs any code is of size $\Omega(\frac{n \log d}{d})$.

Question

Can we always find $H \subseteq G$ satisfying

$$\gamma^{\text{ID}}(H) = O\left(\frac{n \log d}{d}\right)$$

or there is a graph G such that

$$\gamma^{\text{ID}}(H) = \Omega\left(\frac{n \log \Delta}{d}\right)$$

for all $H \subseteq G$?

Open Questions II

Question

Can we apply similar techniques to other non-monotone parameters that behave nicely in random graphs?

THANKS FOR YOUR ATTENTION