Identifying codes in graphs of given maximum degree

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joint works with:

Ralf Klasing, Adrian Kosowski, André Raspaud (2012)
Eleonora Guerrini, Matjaž Kovše, Aline Parreau, Reza Naserasr, Petru Valicov (2011)
Guillem Perarnau (2012)
Sylvain Gravier, Aline Parreau, Reza Naserasr, Petru Valicov (2011+)

Bordeaux Workshop on Identifying Codes
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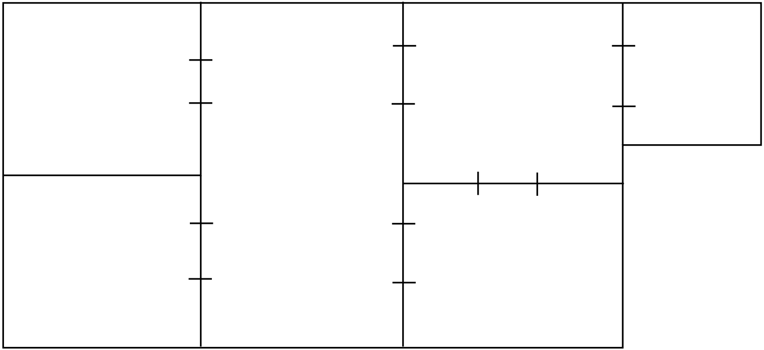
Locating a burglar in a museum



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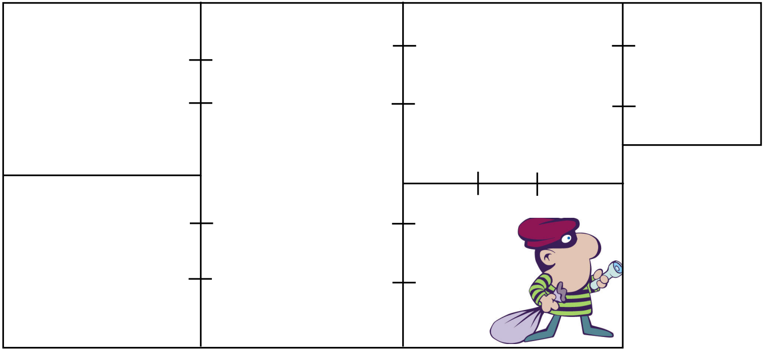

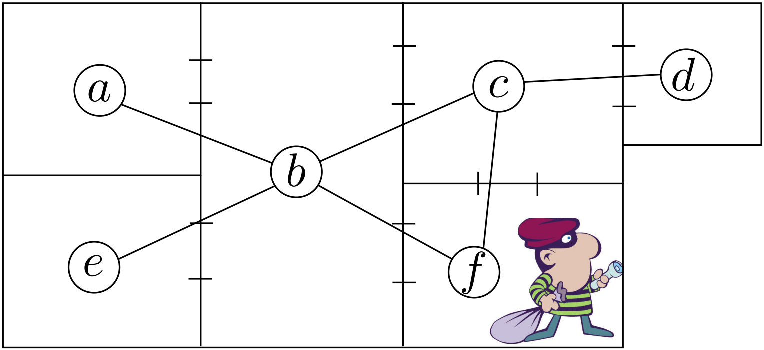

Graph $G = (V, E)$. V: vertices (rooms), $E \subseteq V \times V$: edges (doors)

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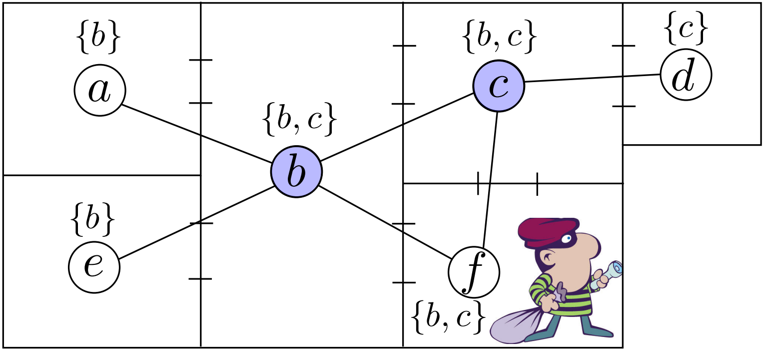

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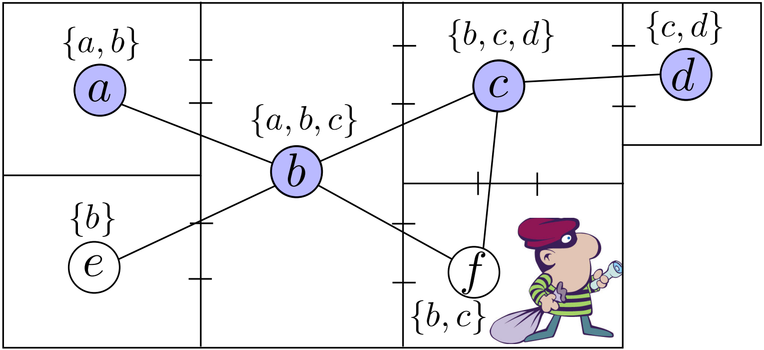

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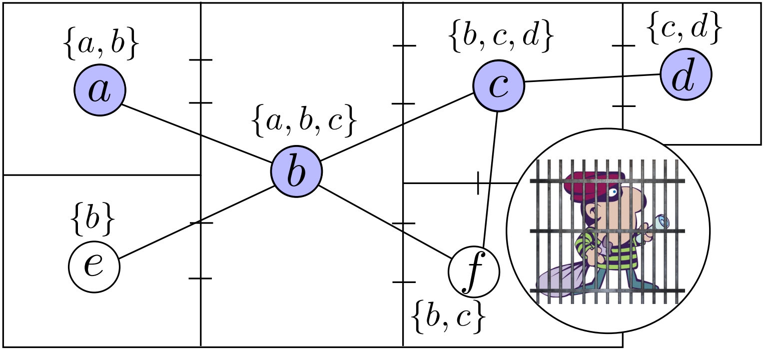

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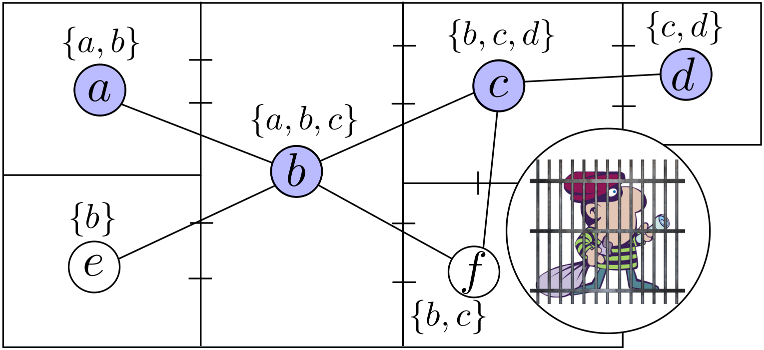

How many detectors do we need?

Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a **dominating set** in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a **separating code** in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

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Proposition

C is an identifying code IFF:

- C is a **dominating set** in G
- $\forall u \neq v$ of V **with** $d_G(u, v) \leq 2$, $(N[u] \Delta N[v]) \cap C \neq \emptyset$

Bounds not related to $\Delta(G)$

Theorem (lower bound: Karpovsky, Chakrabarty, Levitin, 1998
upper bound: Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

Let G be an identifiable graph on n vertices with at least one edge, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

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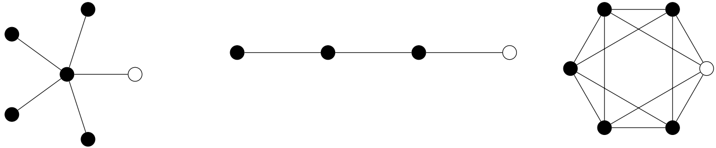
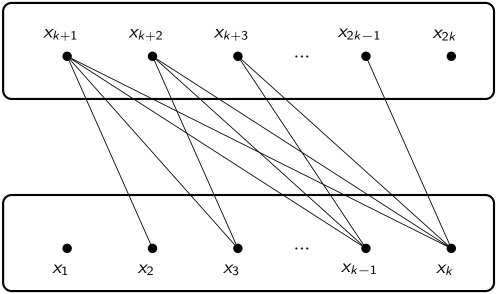


A class of graphs called $\mathcal{A}$

Definition - Graph A_k

$V(A_k) = \{x_1, ..., x_{2k}\}$.
x_i connected to x_j iff $|j - i| \leq k - 1$

Note: $A_1 = \overline{K_2}$; for $k \geq 2$, $A_k = P_{2k}^{k-1}$

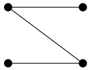


Clique on $\{x_{k+1}, ..., x_{2k}\}$

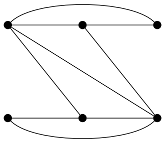
Clique on $\{x_1, ..., x_k\}$

A class of graphs called $\mathcal{A}$ - examples

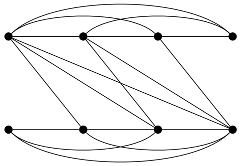
$A_1 = \overline{K_2}$



$A_2 = P_4$



$A_3 = P_6^2$



$A_4 = P_8^3$

A characterization

Definition - Join and its closure

$(\mathcal{A}, \bowtie)$: closure of graphs of $\mathcal{A}$ with respect to $\bowtie$ (complete join).

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

Let G be an identifiable graph on n vertices. Then:

$$\gamma^{\text{ID}}(G) = n - 1 \Leftrightarrow G \in \{K_{1,n-1}\} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1 \text{ and } G \neq \overline{K_2}.$$

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Observation

All these graphs have maximum degree $n-1$ or $n-2$!

A conjecture

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

Let G be an identifiable graph with maximum degree Δ and n vertices, then

$$\frac{2n}{\Delta+2} \leq \gamma^{\text{ID}}(G)$$

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Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected nontrivial identifiable graph on n vertices and of maximum degree Δ. Then:

$$\gamma^{\mathrm{ID}}(G) \leq n - \frac{n}{\Delta} + c \text{ (for some constant } c\text{).}$$

The conjecture is true for $\Delta = 2$ (with $c = 3/2$).

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Extremal examples

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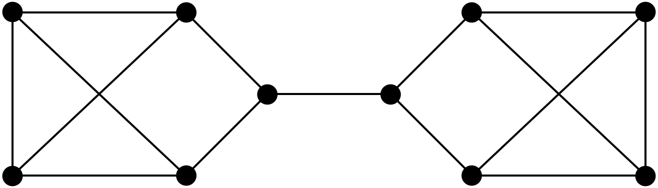

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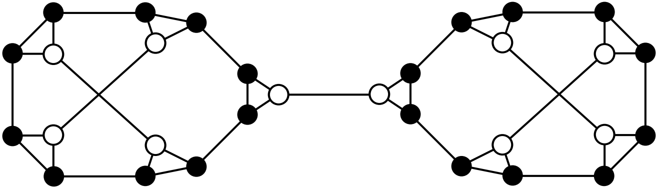

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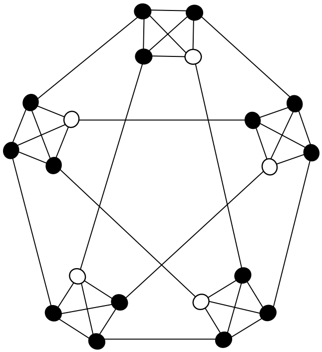

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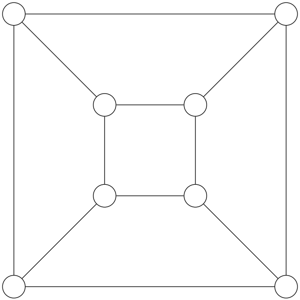

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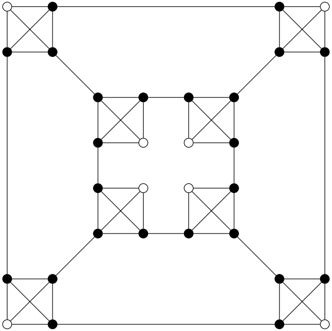

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Also: Sierpiński graphs
(see A. Parreau, S. Gravier, M. Kovše, M. Mollard and J. Moncel, 2011+)

Extremal examples

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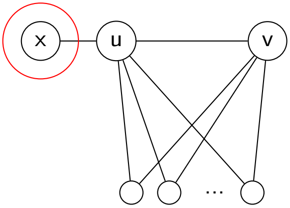
Question

Can we prove that $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta)}$?

u, v such that $N[v] \ominus N[u] = \{x\}$



Then $x \in C$, forced by uv.

Note: if G regular, no forced vertices.

Theorem (F., Guerrini, Kovse, Naserasr, Parreau, Valicov, 2011)

Let G be a connected identifiable graph of maximum degree Δ. Then

$$\gamma^{\mathrm{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^5)}$$

If G is Δ-regular, $\gamma^{\mathrm{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$

Proof idea:

Proposition

Let I be a distance 4-independent set of G. If for all $x \in I$, x is **not forced**, $V - I$ is also an identifying code.

First bounds

Theorem (F., Guerrini, Kovse, Naserasr, Parreau, Valicov, 2011)

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Lemma (Bertrand, Hudry, 2001)

For each vertex x of G, there exists a non forced vertex y in $N[x]$.

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Lemma (Bertrand, Hudry, 2001)

For each vertex x of G, there exists a non forced vertex y in $N[x]$.

Take a (maximal) 6-independent set I. Find the set I' "good vertices" which are not forced: $|I| = |I'|$. $V - I'$ is an identifying code.
For regular graphs, there are no forced vertices: a 4-IS is enough.

Theorem (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected identifiable **triangle-free** graph on n vertices and of maximum degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{(1+\frac{3}{\ln \Delta - 1})\Delta} = n - \frac{n}{(1+o_\Delta(1))\Delta}$$

Proof idea:

Let X be the set of vertices having at least some **false twin** (false twins: $u \not\sim v$ and $N(u) = N(v)$).

- If X is large, at least $\frac{|X|}{\Delta}$ vertices can be out of a code and we are done
- Otherwise, build a maximal independent set S with $|S| > \frac{\ln \Delta}{\Delta} n$ (using J. Shearer's bound)
- Locally modify S to get S', not too small: $|S'| \geq |S|/3$
- $V \setminus S'$ is an identifying code

Triangle-free graphs

Theorem (F., Klasing, Kosowski, Raspaud, 2009)

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Theorem (F., Klasing, Kosowski, Raspaud, 2009)

In fact: let G be a connected identifiable **triangle-free** graph on n vertices and of maximum degree Δ s.t. for all subgraphs H, $\alpha(H) \geq f(\Delta) n_H$. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta + \frac{3}{f(\Delta)}}$$

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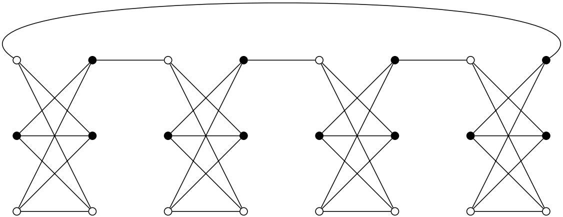
$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta + \frac{3}{f(\Delta)}}$$

Corollary

G k-colourable: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+3k}$.
$\Rightarrow$ Bipartite: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+6}$
$\Rightarrow$ Planar triangle-free: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+9}$

Complete $(\Delta - 1)$-ary tree, caterpillar: roughly, $\gamma^{\text{ID}}(G) = n - \frac{n}{\Delta - 1}$



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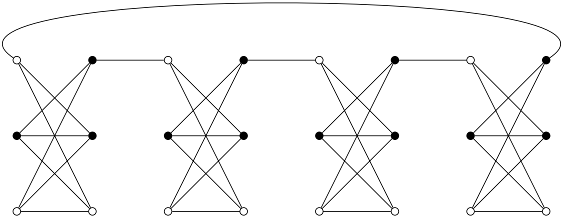

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Question

What about triangle-free graphs without false twins?

Notation

Let $NF(G)$ be the proportion of **non** forced vertices of G

$$NF(G) = \frac{\#\text{non-forced vertices in G}}{\#\text{vertices in G}}$$

Theorem (F., Perarnau, 2011)

For each identifiable graph G on n vertices having maximum degree $\Delta \geq 3$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{103\Delta}$$

Proof idea:

- Take all forced vertices (set F) into the code.
- From $V \setminus F$, select each vertex with probability $p_S = \frac{1}{k \cdot \Delta}$ (k constant) to belong to a set S. We want $C = V \setminus S$.
- Use Lovász' Local Lemma to show that $\Pr(C \text{ is a code}) > f(k, n, \Delta) > 0$
- Use the Chernoff bound to show that $\Pr(C \text{ is too small}) < f(k, n, \Delta)$

Bounding the number of forced vertices

Proposition

$$\frac{1}{\Delta+1} \leq NF(G) \leq 1$$

Proof:

Lemma (Bertrand, Hudry, 2001)

Let G be an identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex y in $N[x]$.

$\Rightarrow$ The set S of non-forced vertices forms a dominating set. Hence $|S| \geq \frac{n}{\Delta+1}$.

Bounding the number of forced vertices

Proposition

Let G be a graph of clique number at most k. There exists a function ρ such that:

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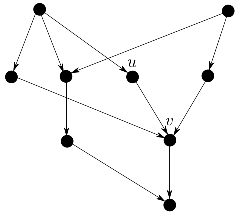
- Define graph $\overrightarrow{H}(G)$
- Max. degree of $\overrightarrow{H}(G)$: $2k-3$
- Longest directed chain of $\overrightarrow{H}(G)$: $k-1$
- Each component has a non-forced vertex
- $\Rightarrow \rho(k) \leq \sum_{i=0}^{k-2}(2k-3)^i$



$u \rightarrow v$ if $N[v] = N[u] \cup \{x\}$

Corollaries

Theorem (F., Perarnau, 2011)

For each identifiable graph G on n vertices having maximum degree $\Delta \geq 3$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{103\Delta}$$

Corollary

- In general, $NF(G) \geq \frac{1}{\Delta+1}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$
- If G is Δ-regular, $NF(G) = 1$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{103\Delta} = n - \frac{n}{\Theta(\Delta)}$
- If G has clique number bounded by k, $NF(G) \geq \frac{1}{\rho(k)}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{103 \cdot (\rho(k))^2 \cdot \Delta} = n - \frac{n}{\Theta(\Delta)}$

Note: for $k = 2, 3, 4, 5$: $103 \cdot (\rho(k))^2 = 103,\ 1.360,\ 81.685,\ 13.600.000$

The conjecture holds for some large subclass of line graphs:

Theorem (F., Gravier, Naserasr, Parreau, Valicov, 2011)

Let G be an edge-identifiable graph with a minimal edge-identifying code C_E. Then $G[C_E]$ is 2-degenerate.

Corollary

If G edge-identifiable, $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq 2|V(G)| - 3$.

Corollary

If G is an edge-identifiable graph **with average degree** $\overline{d}(G) \geq 5$, then $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq n - \frac{n}{\Delta(\mathcal{L}(G))}$ where $n = |V(\mathcal{L}(G))|$.

Questions

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected nontrivial identifiable graph on n vertices and of maximum degree Δ. Then:

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + c \text{ (for some constant } c\text{)}.$$

- Can we reduce the constants?
- Can we improve the bound $n - \frac{n}{\Theta(\Delta^3)}$?
- What about $\Delta = 3$?
- What about trees (having a look at David Auger's algorithm)?
- What about claw-free graphs? $n - \frac{n}{\Theta(\Delta^2)}$ seems to hold by directly using similar arguments than for triangle-free graphs.
- Other related parameters?