Bounding K_4-minor-free graphs in the homomorphism order

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Extended Abstract

A *homomorphism* of a graph G to a graph H is a mapping f from $V(G)$ to $V(H)$ such that if two vertices are adjacent in G, their images by f are adjacent in H. A class $\mathcal{C}$ of graphs is said to be *bounded* by some graph H if each graph of $\mathcal{C}$ admits a homomorphism to H; H is called a *bound* for $\mathcal{C}$. It is of interest to ask for a bound having specicic properties (e.g. having specific odd-girth) with smallest possible order. Such questions are studied e.g. in [1].

The projective cube of dimension $2k$, denoted $PC(2k)$, is the graph obtained from the hypercube of dimension $2k+1$ by identifying each pair of antipodal vertices. $PC(2k)$ has 2^{2k} vertices and odd-girth $2k+1$ (the odd-girth of a graph is the length of one of its shortest odd cycles). For example, $PC(2)$ is K_4 and $PC(4)$ is the Clebsch graph.

The following question was asked by R. Naserasr in [2]:

Problem 1 *Given two integers $r \geq k$, what is the smallest subgraph of $PC(2k)$ to which every planar graph of odd-girth $2r+1$ admits a homomorphism to?*

R. Naserasr [2] showed that this question is related to many important theories and captures problems such as edge-colouring, fractional colouring, and circular colouring planar graphs. Motivated by this question, we study the analogous question for the class of series-parallel graphs, i.e. K_4-minor-free graphs. Let $\mathcal{SP}_{2k+1}$ be the class of series-parallel graphs of odd-girth at least $2k+1$. We prove:

Theorem 2 *$PC(2k)$ is a bound for $\mathcal{SP}_{2k+1}$.*

We use the previous theorem to give a reformulation of the question of finding a bound of odd-girth $2k+1$ of smallest order for $\mathcal{SP}_{2k+1}$. We present partial answers for the first three cases:

Theorem 3 *The triangle K_3, the Wagner graph and the graph G_{16} of Figure 1 are bounds for $\mathcal{SP}_3$, $\mathcal{SP}_5$ and $\mathcal{SP}_7$, respectively.*

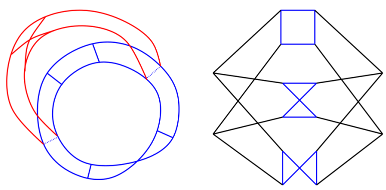

Figure 1: Two drawings of the graph G_{16} that is a bound for $\mathcal{SP}_7$.

References

[1] T. H. Marshall, R. Naserasr and J. Nešetřil. Homomorphism bounded classes of graphs. **Eur. J. Combin.** 27(4):592–600, 2006.

[2] R. Naserasr. Mapping planar graphs into projective cubes. To appear in **J. Graph Theor.**