Information Processing Letters 170 (2021) 106121

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

# Complexity and algorithms for injective edge-coloring in graphs

Florent Foucaud [a,b,c,*], Hervé Hocquard [b], Dimitri Lajou [b]

[a] LIMOS, CNRS UMR 6158, Université Clermont Auvergne, Aubière, France
[b] Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France
[c] Univ. Orléans, INSA Centre Val de Loire, LIFO EA 4022, F-45067 Orléans, France

ARTICLE INFO

*Article history:*
Received 2 November 2020
Received in revised form 30 March 2021
Accepted 30 March 2021
Available online xxxx
Communicated by S. Albers

*Keywords:*
Injective edge-coloring
Planar graphs
Subcubic graphs
Graph algorithms
Treewidth

ABSTRACT

An injective $k$-edge-coloring of a graph $G$ is an assignment of colors, i.e. integers in $\{1, \dots, k\}$, to the edges of $G$ such that any two edges each incident with one distinct endpoint of a third edge, receive distinct colors. The problem of determining whether such a $k$-coloring exists is called Injective $k$-Edge-Coloring. We show that Injective 3-Edge-Coloring is NP-complete, even for triangle-free cubic graphs, planar subcubic graphs of arbitrarily large girth, and planar bipartite subcubic graphs of girth 6. Injective 4-Edge-Coloring remains NP-complete for cubic graphs. For any $k \geq 45$, we show that Injective $k$-Edge-Coloring remains NP-complete even for graphs of maximum degree at most $5\sqrt{3k}$. In contrast with these negative results, we show that Injective $k$-Edge-Coloring is linear-time solvable on graphs of bounded treewidth. Moreover, we show that all planar bipartite subcubic graphs of girth at least 16 are injectively 3-edge-colorable. In addition, any graph of maximum degree at most $\sqrt{k/2}$ is injectively $k$-edge-colorable.



## 1. Introduction

We study the algorithmic complexity of the injective edge-coloring problem. Our aim is to determine restricted graph classes where the problem is NP-hard, while in contrast, designing algorithms for other graph classes. An *injective k-edge-coloring* of a graph $G = (V(G), E(G))$ is an assignment of colors, i.e. integers in $\{1, \dots, k\}$, to the edges of $G$ in such a way that two edges that are each incident with one distinct endpoint of a third edge, receive distinct colors. In other words, for any 3-edge path of $G$ (possibly forming a triangle), the first and last edge of the path receive distinct colors. The *injective chromatic index* of $G$, denoted $\chi_i'(G)$, is the least integer $k$ for which $G$ admits an injective $k$-edge-coloring.

This concept was recently introduced in [4], where it is studied for some classes of graphs, and proved to be NP-complete. Bounds on the injective chromatic index of planar graphs, graphs of given maximum degree, and other important graph classes, have been recently determined in [1,3,7,14,16]. In particular, as mentioned in [7], it follows from [1] that all planar graphs are injectively 30-edge-colorable, while outerplanar graphs are injectively 9-edge-colorable [7]. It is also proved in [14] that subcubic graphs are injectively 7-edge-colorable, while subcubic bipartite graphs [7] and subcubic planar graphs [14] are injectively 6-edge-colorable. Moreover all subcubic planar bipartite graphs are injectively 4-edge-colorable [14].

Note that in [1], this notion is studied as the *induced star arboricity* of a graph, that is, the smallest number of star forests into which the edges of the graph can be partitioned: this is an equivalent way to interpret injective edge-coloring (see [7]). The concept of an injective edge-coloring is the natural edge-version of the notion of

* Corresponding author.
*E-mail address:* florent.foucaud@uca.fr (F. Foucaud).

https://doi.org/10.1016/j.ipl.2021.106121
0020-0190/

an injective vertex-coloring, introduced in [10] and well-studied since then.

Another closely related notion is the one of *strong edge-coloring* of a graph $G$, introduced in [8] and well-studied since then, especially in view of a celebrated conjecture by Erdős and Nešetřil [6]. In this type of coloring, edges that are the endpoints of a same 3-edge path *or 2-edge path* must receive distinct colors. The *strong chromatic index* $\chi'_s(G)$ of a graph $G$ is the least integer $k$ for which $G$ admits a strong edge-coloring with $k$ colors. It follows from the definitions that for any graph $G$, $\chi'_i(G) \leq \chi'_s(G)$ holds.

The algorithmic complexity of determining the strong chromatic index of a graph is well-studied, see for example [12] for a classic reference, and [5,11] for more recent ones. In this paper, we wish to undertake similar types of studies for the injective chromatic index. The problem at hand is formally defined as follows.

Injective $k$-Edge-Coloring
Instance: A graph $G$.
Question: Does $G$ admit an injective $k$-edge-coloring?

Injective $k$-Edge-Coloring was proved NP-complete (for every fixed $k \geq 3$) in [4], with no particular restriction on the inputs. We strengthen this as follows.

**Theorem 1.** *The two following are NP-Complete:*

1. Injective 3-Edge-Coloring, *even for triangle-free cubic graphs, and*
2. Injective 4-Edge-Coloring, *even for cubic graphs.*

Answering a question from [4] about the complexity of Injective $k$-Edge-Coloring for planar graphs, we also study restricted subclasses of planar graphs.

**Theorem 2.** *Let* $g \geq 3$. Injective 3-Edge-Coloring *is NP-Complete even for:*

1. *planar subcubic graphs with girth at least g,*
2. *planar bipartite subcubic graphs of girth 6.*

The two items in Theorem 2 cannot be combined, because we can prove the following (note that all planar bipartite subcubic graphs are injectively 4-edge-colorable [14]).

**Theorem 3.** *Every planar bipartite subcubic graph of girth at least 16 is injectively 3-edge-colorable.*

We also obtain the following positive result ($tw(G)$ denotes the treewidth of $G$).

**Theorem 4.** *For every graph G of order n and every positive integer k, there exists a* $2^{O(k \cdot tw(G)^2)} n$*-time algorithm that solves* Injective $k$-Edge-Coloring.

It is proved in [1] that $\chi'_i(G) \leq 3\binom{tw(G)}{2}$, and so using the above algorithm, one can determine the injective chromatic index of a graph of order $n$ in time $2^{O(tw(G)^4)} n$.

Contrasting with our hardness results for planar graphs, Theorem 4 implies that Injective $k$-Edge-Coloring can be solved in polynomial-time on subclasses of planar graphs: $K_4$-minor-free graphs (i.e. graphs of treewidth 2), and thus, on outerplanar graphs.

In [4], Cardoso *et al.* use a reduction on graphs having their maximum degree linear in the number of colors. We improve it with the following result.

**Theorem 5.** *For every integer* $k \geq 45$, Injective $k$-Edge-Coloring *is NP-Complete even for graphs with maximum degree at most* $5\sqrt{3k}$.

The bound of Theorem 5 is tight up to a constant factor: by a standard maximum degree argument of a conflict graph, every graph with maximum degree at most $\sqrt{k/2}$ is injectively $k$-edge-colorable. (Indeed, for every edge $e$ of a graph $G$, there are at most $2(\Delta(G)-1)^2$ edges which cannot have the same color as $e$, where $\Delta(G)$ is the maximum degree of $G$.)

## 2. Proof of Theorem 1

For these two problems, we reduce from 3-Edge-Coloring, which is NP-Complete even for cubic graphs [12]. (Recall that a proper edge-coloring is an edge-coloring for which edges that are incident to a same vertex receive different colors.)

3-Edge-Coloring
Instance: A cubic graph $G$.
Question: Does $G$ admit a proper 3-edge-coloring?

### 2.1. Proof of Theorem 1.1

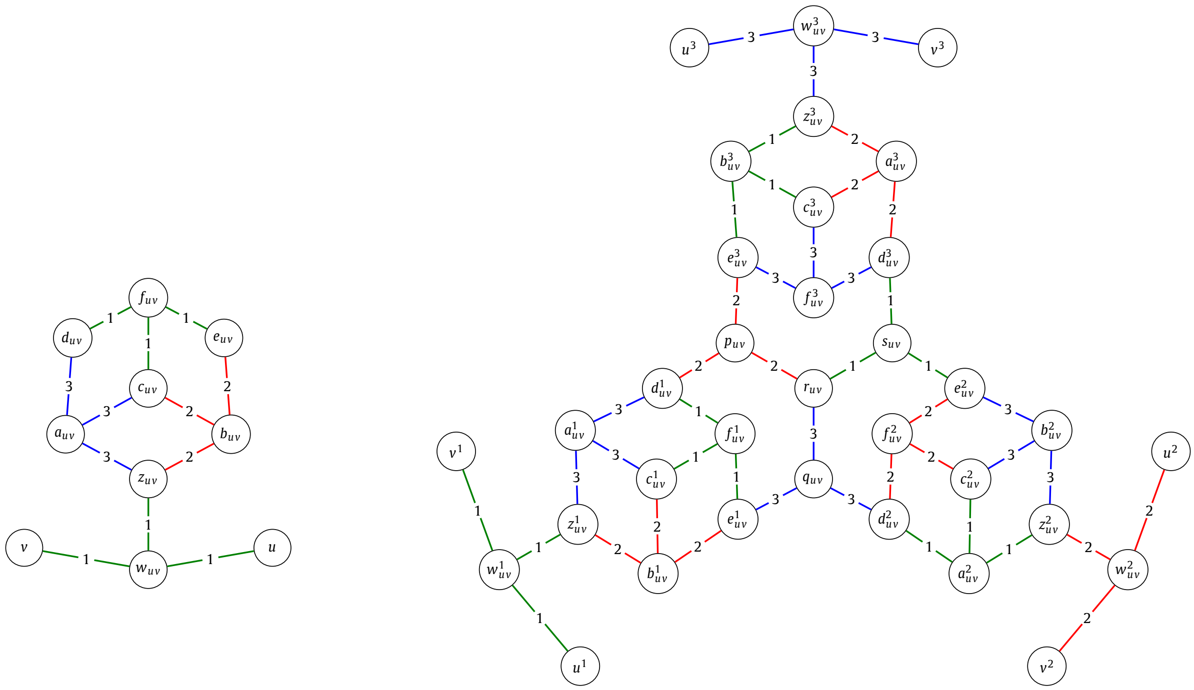
**Proof.** Let $G$ be the input cubic graph. We will proceed in two steps: first, we create a triangle-free subcubic graph $G'$ which has an injective 3-edge-coloring if and only if $G$ is properly 3-edge-colorable. Then we describe how to make the graph cubic.

We create the graph $G'$ from $G$ by removing all the edges of $G$. For each edge $uv$ of $G$, we create a copy of a gadget $E_{uv}$ (see Fig. 1(a) for an illustration) and connect it to $u$ and $v$ as follows. We add eight new vertices $w_{uv}$, $z_{uv}$, $a_{uv}$, $b_{uv}$, $c_{uv}$, $d_{uv}$, $e_{uv}$ and $f_{uv}$. We create the following edges $uw_{uv}$, $vw_{uv}$, $w_{uv}z_{uv}$, $z_{uv}a_{uv}$, $z_{uv}b_{uv}$, $a_{uv}c_{uv}$, $b_{uv}c_{uv}$, $a_{uv}d_{uv}$, $b_{uv}e_{uv}$, $c_{uv}f_{uv}$, $d_{uv}f_{uv}$ and $e_{uv}f_{uv}$.

**Claim 6.** *$E_{uv}$ is injectively 3-edge-colorable, and for every valid edge-coloring $\gamma$ of $E_{uv}$, $\gamma(uw_{uv}) = \gamma(vw_{uv}) = \gamma(w_{uv}z_{uv})$. Moreover, for any choice of the same color for these three edges, we can extend the coloring to an injective 3-edge-coloring of $E_{uv}$.*

**Proof.** Let us injectively 3-edge-color $E_{uv}$. W.l.o.g., we can assume that $d_{uv}f_{uv}$ is colored 1, $b_{uv}c_{uv}$ is colored 2 and $a_{uv}z_{uv}$ is colored 3. We deduce that $b_{uv}e_{uv}$ is colored 2, $c_{uv}f_{uv}$ is colored 1, $a_{uv}d_{uv}$ and $a_{uv}c_{uv}$ are colored 3, $b_{uv}z_{uv}$ is colored 2 and $e_{uv}f_{uv}$ is colored 1. Hence $uw_{uv}$, $vw_{uv}$ and $w_{uv}z_{uv}$ must all be colored 1.

Now, given one same color for these three edges, one can color the rest of the gadget, for example using the previously constructed coloring. ■




(a) Edge gadget $E_{uv}$ with an injective 3-edge-coloring.

(b) Connecting three copies of $E_{uv}$ in the construction of $G''$, along with an injective 3-edge-coloring.

**Fig. 1.** Edge gadgets used in the proof of Theorem 1.1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

If $G$ has a proper 3-edge-coloring $\gamma$, we injectively 3-edge-color $G'$ by assigning to $uw_{uv}$, $vw_{uv}$ and $w_{uv}z_{uv}$ in $G'$ the color $\gamma(uv)$; then we extend the coloring to each $E_{uv}$ using Claim 6.

Conversely, if $G'$ has an injective 3-edge-coloring, then we color an edge $uv$ of $G$ with the color of the edge $uw_{uv}$ (or $vw_{uv}$) of $G'$. This coloring is proper since Claim 6 insures that $uw_{uv}$ and $vw_{uv}$ have the same color. Indeed if $ux$ is an edge adjacent to $uv$, then $uw_{uv}$ and $xw_{ux}$ have different colors.

We now show how to make the construction cubic. We create the cubic graph $G''$ as follows. First, take three disjoint copies $G_1$, $G_2$ and $G_3$ of $G'$. To differentiate the vertices of each copy, we add an exponent to the name of the vertex corresponding to the number of the copy. For example, vertex $w_{uv}$ of $G_1$ will be noted $w^1_{uv}$. For each edge $uv$ of $G$, connect $G_1$, $G_2$ and $G_3$ via $K_{1,3}$ with vertex classes $\{r_{uv}\}$ and $\{s_{uv}, p_{uv}, q_{uv}\}$ as follows. The vertex $s_{uv}$ (resp. $p_{uv}$, resp. $q_{uv}$) is adjacent to $d^3_{uv}$ (resp. $d^1_{uv}$, resp. $d^2_{uv}$), $e^2_{uv}$ (resp. $e^3_{uv}$, resp. $e^1_{uv}$) and $r_{uv}$ (see Fig. 1(b)). The graph $G''$ is simply the graph where the edge gadget is represented in Fig. 1 and for each $u \in V(G)$, the three copies of $u^i$ for $i \in \{1, 2, 3\}$ are identified.

As $G$ is cubic, $G''$ is triangle-free and cubic. Note that if $G''$ admits an injective 3-edge-coloring, then in particular $G'$ also admits an injective 3-edge-coloring and thus by our previous arguments, $G$ is properly 3-edge-colorable.

If $G$ is properly 3-edge-colorable, then we fix such a coloring $\gamma : E(G) \to \{1, 2, 3\}$. For $i \in \{1, 2, 3\}$, we color the edges incident with $w^i_{uv}$ with the color $\gamma(uv) + i$, where the colors are considered to be taken modulo 3 (considering $0 = 3$). Then it suffices to extend the obtained coloring to each edge gadget (see Fig. 1). $\square$

### 2.2. Proof of Theorem 1.2

**Proof.** Let $G$ be the input graph. For each vertex $u$ of $G$, we replace it by the following vertex gadget $S_u$ (see Fig. 2). The gadget $S_u$ is made of a 9-cycle $x^u_0 x^u_1 \dots x^u_8$ and three other vertices $y^u_i$ ($i \in \{0, 3, 6\}$) that will be connected to the rest of the graph. We add the edges $x^u_1 x^u_8$, $x^u_2 x^u_4$, $x^u_5 x^u_7$, $x^u_0 y^u_0$, $x^u_3 y^u_3$ and $x^u_6 y^u_6$. For any edge-coloring $\gamma$ of $S_u$, we note $C^u_i(\gamma) = \left\{\gamma(x^u_i x^u_{i+1}), \gamma(x^u_i x^u_{i-1})\right\}$ where $i \in \{0, 3, 6\}$ and where the indices are taken modulo 9.

**Claim 7.** *For every injective 4-edge-coloring $\gamma$ of $S_u$ and for every $i \in \{0, 3, 6\}$, the color $\gamma(x^u_i y^u_i)$ belongs to the set $C^u_i(\gamma)$. Moreover, $C^u_0(\gamma) \cup C^u_3(\gamma) \cup C^u_6(\gamma) = \{1, 2, 3, 4\}$ and there exists a color $a \in \{1, 2, 3, 4\}$ such that for all $i \in \{0, 3, 6\}$, $a \in C^u_i(\gamma)$.*

*Furthermore, for any choice of color for $x^u_0 y^u_0$, $x^u_3 y^u_3$, $x^u_6 y^u_6$ and sets of colors $C^u_i(\gamma)$, $i \in \{0, 3, 6\}$ verifying the previous necessary conditions, there exists an injective 4-edge-coloring $\gamma$ of $S_u$ matching those choices.*

**Proof.** Let us try to construct an injective 4-edge-coloring $\gamma$ of $S_u$. Up to permuting the colors, we assume that $\gamma(x^u_0 x^u_1) = 1$, $\gamma(x^u_0 x^u_8) = 2$ and $\gamma(x^u_8 x^u_1) = 3$. Note that $x^u_2 x^u_4$ and $x^u_5 x^u_7$ cannot both be colored 4, w.l.o.g. assume that

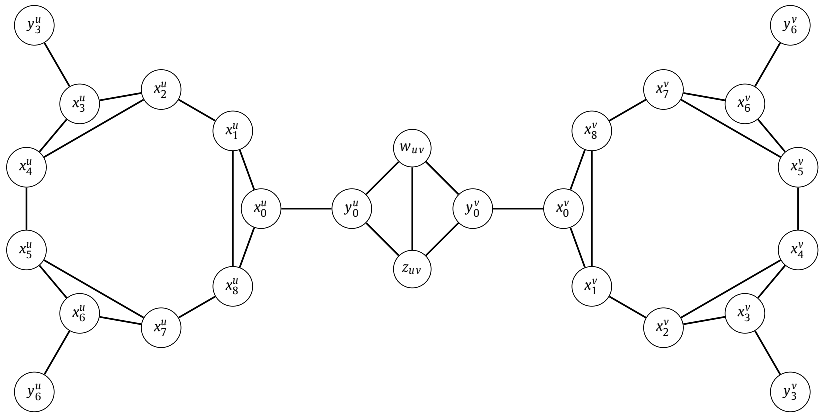



**Fig. 2.** Two vertex gadgets $S_u$ and $S_v$, corresponding to the vertices $u$ and $v$ of a graph $G$, connected by an edge gadget corresponding to the edge $uv$ of $G$.

$\gamma(x_2^u x_4^u) \neq 4$. Hence $\gamma(x_2^u x_4^u) = 2$ and $\gamma(x_2^u x_3^u) = 4$. Remark that $\gamma(x_5^u x_6^u) \neq 2$. Moreover $x_5^u x_7^u$ and $x_6^u x_7^u$ can only receive colors 1 or 4 and they must receive different colors. Hence $\gamma(x_5^u x_6^u) = 3$, $\gamma(x_3^u x_4^u) = 1$, $\gamma(x_5^u x_7^u) = 4$ and $\gamma(x_6^u x_7^u) = 1$. Now there are two ways to complete the coloring of $S_u$, either $\gamma(x_1^u x_2^u) = 4$, $\gamma(x_4^u x_5^u) = 3$ and $\gamma(x_7^u x_8^u) = 2$ or, $\gamma(x_1^u x_2^u) = 3$, $\gamma(x_4^u x_5^u) = 2$ and $\gamma(x_7^u x_8^u) = 4$. In both cases all properties of the first part of the claim hold (with $a = 1$).

Finally, note that the second of the two previous coloring options allows us to color $x_i^u y_i^u$, $i \in \{0, 3, 6\}$ with any color among those of $x_i^u x_{i+1}^u$ and $x_i^u x_{i-1}^u$, and to complete the coloring. ■

For every edge $uv$ of $G$, we construct the following edge gadget $E_{uv}$ (see Fig. 2). First, choose $y_i^u$ (resp. $y_j^v$) of degree 1 among the vertices of $S_u$ (resp. $S_v$). Create two new adjacent vertices $w_{uv}$ and $z_{uv}$ such that $y_i^u w_{uv} y_j^v z_{uv}$ is a 4-cycle.

**Claim 8.** *For every injective 4-edge-coloring $\gamma$ of $G$ and every edge gadget $E_{uv}$ connecting $y_i^u$ and $y_j^v$ ($i, j \in \{0, 3, 6\}$), we have $C_i^u(\gamma) = C_j^v(\gamma)$.*

*Furthermore, any injective 4-edge-coloring $\gamma$ of $S_u$ and $S_v$ such that $C_i^u(\gamma) = C_j^v(\gamma)$ and $\gamma(x_i^u y_i^u) = \gamma(x_j^v y_j^v)$ can be extended to an injective 4-edge-coloring of $S_u \cup E_{uv} \cup S_v$.*

**Proof.** Suppose, w.l.o.g. by Claim 7, that $x_i^u x_{i+1}^u$ is colored 1, $x_i^u x_{i-1}^u$ is colored 2 and $x_i^u y_i^u$ is colored 1. Now w.l.o.g., $y_i^u w_{uv}$ is colored 3 and $y_i^u z_{uv}$ is colored 4. This implies that $w_{uv} z_{uv}$ is colored 2, $y_j^v w_{uv}$ is colored 3, $y_j^v z_{uv}$ is colored 4, $y_j^v x_j^v$ is colored 1 and $C_j^v(\gamma) = \{1, 2\}$.

The second part of the claim is proved by taking the previous coloring and extending it using the second part of Claim 7. ■

Let $G'$ be the cubic graph constructed from $G$ by the above process. By Claim 8, if $uv$ is an edge connecting $y_i^u$ and $y_j^v$ then for any injective coloring $\gamma$ of $G'$, $C_i^u(\gamma) = C_j^v(\gamma) = \{a, b\}$ for some $a$ and $b$. Hence this set somehow characterizes the edge gadget $E_{uv}$, we say that $E_{uv}$ is *colored* by $\{a, b\}$.

Suppose that there exists an injective 4-edge-coloring $\gamma$ of $G'$. For each edge $uv$ of $G$, we color $uv$ depending on the coloring of $E_{uv}$. When $E_{uv}$ is colored $\{1, 2\}$ or $\{3, 4\}$ (resp. $\{1, 3\}$ or $\{2, 4\}$, resp. $\{1, 4\}$ or $\{2, 3\}$) then we color $uv$ by color 1 (resp. 2, resp. 3). We argue that this edge-coloring, noted $\gamma$, is proper. Indeed suppose it is not, then for some vertex $u$, w.l.o.g., $uv$ and $uw$ are both colored 1. This means that the coloring of $G'$ is such that $C_i^u(\gamma) = C_j^u(\gamma)$ or $C_i^u(\gamma) \cap C_j^u(\gamma) = \varnothing$ for $i \neq j$ and $i, j \in \{0, 3, 6\}$. This contradicts Claim 7. Hence we get a proper 3-edge-coloring of $G$.

Conversely, suppose that there exists a proper 3-edge-coloring of $G$. In $G'$, we color each edge of the form $x_i^u y_i^u$ by 1. If an edge $uv$ of $G$ is colored 1 (resp. 2, resp. 3) then we assign the color $\{1, 2\}$ (resp. $\{1, 3\}$, resp. $\{1, 4\}$) to $E_{uv}$. By Claim 7, this coloring can be extended to an injective 4-edge-coloring of each $S_u$, $u \in V(G)$. By Claim 8, this injective 4-edge-coloring can be extended to each edge gadget to color the whole graph. □

## 3. Proof of Theorem 2

We will reduce from the following problem:

PLANAR 3-VERTEX-COLORING
Instance: A planar graph $G$ with maximum degree 4.
Question: Does $G$ admit a proper 3-vertex-coloring?

This problem was proven to be NP-Complete in [9]. Let $G$ be a planar graph with maximum degree 4.

### 3.1. *Proof of Theorem 2.1*

**Proof.** Recall that we want to construct a graph $G'$ with girth at least $g$.

For each vertex $u \in V(G)$, we construct a vertex gadget $S_u$ as follows (see Fig. 3). First create a cycle $x_1^u x_2^u \dots x_\ell^u$ where $\ell \geq g$ and $\ell$ is an odd multiple of 3. To each $x_i^u$ add a single pendant neighbor $y_i^u$ of degree 1. To the vertex $y_1^u$, add two non-adjacent neighbors $w^u$ and $z^u$. Create four more vertices $a_1^u$, $b_1^u$, $c_1^u$ and $d_1^u$. The vertex $w^u$ is adjacent to $a_1^u$ and $b_1^u$ while $z^u$ is adjacent to $c_1^u$ and $d_1^u$. Now construct a path $a_1^u a_2^u \dots a_g^u$ of length $g$ and add to

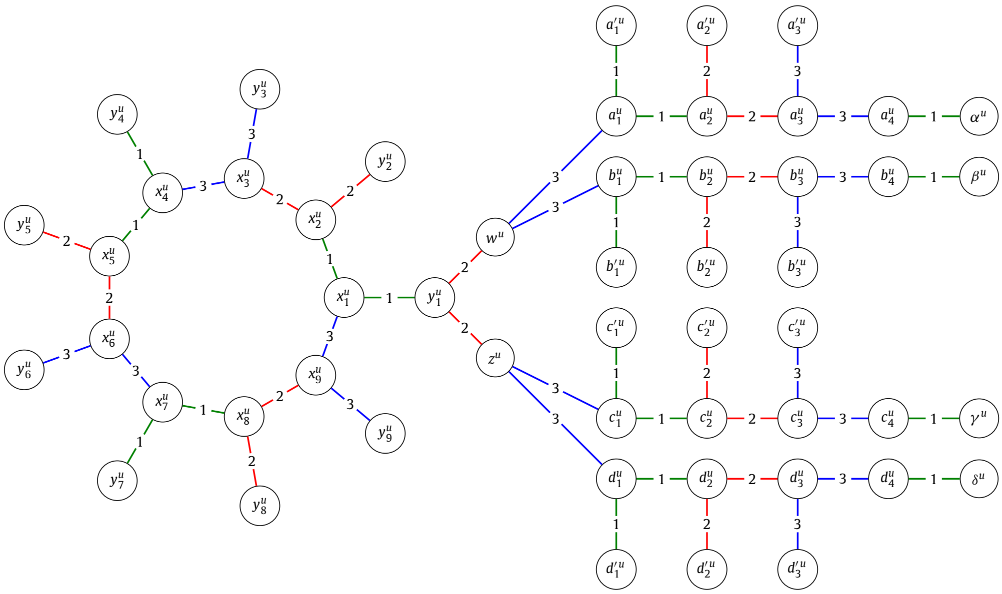


**Fig. 3.** Vertex gadget $S_u$ for planar subcubic graphs with girth at least $g$ (in this example $g=4$ and $\ell=9$).

each $a_i^u$ for $i \le g-1$ a pendant vertex of degree 1 called $a_i'^u$. Similarly we create the vertices $b_1^u \dots b_g^u, b_1'^u \dots b_{g-1}'^u$, $c_1^u \dots c_g^u, c_1'^u \dots c_{g-1}'^u$ and $d_1^u \dots d_g^u, d_1'^u \dots d_{g-1}'^u$. Finally add a vertex $\alpha^u$ (resp. $\beta^u$, resp. $\gamma^u$, resp. $\delta^u$) adjacent to $a_g^u$ (resp. $b_g^u$, resp. $c_g^u$, resp. $d_g^u$).

**Claim 9.** *For any injective* 3*-edge-coloring* $\rho$ *of* $S_u$, $\rho(a_g^u\alpha^u) = \rho(b_g^u\beta^u) = \rho(c_g^u\gamma^u) = \rho(d_g^u\delta^u)$. *We call this color* $\rho(S_u)$. *Moreover, for any choice of a color* $\rho(S_u)$*, there exists an injective* 3*-edge-coloring* $\rho$ *with these properties.*

**Proof.** Suppose that there exists $i \in \{1,\dots,\ell\}$ such that the property $\mathcal{P}(i) :=$ "$\rho(x_i^u x_{i+1}^u) = \rho(x_i^u y_i^u) \neq \rho(x_i^u x_{i-1}^u)$" holds (the indices are taken modulo $\ell$, considering $0=\ell$). Then $\mathcal{P}(i)$ holds for all $i \in \{1,\dots,\ell\}$. Indeed, take such an $i$, then $\rho(x_{i+1}^u x_{i+2}^u) = \rho(x_{i+1}^u y_{i+1}^u)$ is the color $\{1,2,3\} \setminus \left\{\rho(x_i^u y_i^u), \rho(x_i^u x_{i-1}^u)\right\}$. Hence the property holds for $i+1$, by induction it holds for every $i$. Note that the same can be said for the property $\mathcal{P}'(i) =$ "$\rho(x_i^u x_{i-1}^u) = \rho(x_i^u y_i^u) \neq \rho(x_i^u x_{i+1}^u)$". Also note that if $\rho(x_i^u x_{i-1}^u) = \rho(x_i^u x_{i+1}^u) \neq \rho(x_i^u y_i^u)$ then we have $\mathcal{P}(i+1)$ which is a contradiction because we do not have $\mathcal{P}(i)$.

Suppose now that for all $i$, neither $\mathcal{P}(i)$ nor $\mathcal{P}'(i)$ holds. This means that the edges incident to a vertex $x_i^u$ are either of the same color, or of three distinct colors. If they have the same color, then the edges incident with $x_{i+1}^u$ have three distinct colors, the ones incident to $x_{i+2}^u$ have the same color, and so on. This would imply that the cycle $x_1^u \dots x_\ell^u$ is even, which is a contradiction. Moreover, if the edges incident to $x_i^u$ have three distinct colors, then the edges incident to $x_{i+1}^u$ (or $x_{i-1}^u$) would all have the same color, and therefore no injective 3-edge-coloring would be possible.

Thus, w.l.o.g. we can suppose that $\rho(x_1^u x_2^u) = \rho(x_1^u y_1^u) = 1$ and $\rho(x_1^u x_\ell^u) = 3$. By extending the coloring to the rest of $S_u$, we can infer that $\rho(y_1^u w^u) = \rho(y_1^u z^u) = 2$, $\rho(w^u a_1^u) = \rho(w^u b_1^u) = 3$ and $\rho(z^u c_1^u) = \rho(z^u d_1^u) = 3$. By the same reasoning, we can see that all the edges of $S_u$ (ignoring the edges involving one of the vertices $x_i^u$) have only one possible color which depends only on their distance to $y_1^u$ and in particular $\rho(a_g^u\alpha^u) = \rho(b_g^u\beta^u) = \rho(c_g^u\gamma^u) = \rho(d_g^u\delta^u)$.

Conversely, $S_u$ admits a coloring (see Fig. 3 for an example). To choose a coloring of $S_u$ having the desired color $\rho(S_u)$, it suffices to permute the colors in the previous coloring. ■

To finish the construction, for any edge $uv \in E(G)$, we add an edge $e^{uv}$ to $G'$ between a vertex among $\left\{\alpha^u, \beta^u, \gamma^u, \delta^u\right\}$ and a vertex among $\left\{\alpha^v, \beta^v, \gamma^v, \delta^v\right\}$ such that the planarity of $G'$ is preserved. This can be done by cyclically ordering the vertices of $\left\{\alpha^u, \beta^u, \gamma^u, \delta^u\right\}$ according to a planar embedding of $G$, and adding the edge $e^{uv}$ between the right pair of vertices.

Note that $G'$ is planar, subcubic with girth at least $g$.

Suppose that $G'$ admits an injective 3-edge-coloring $\rho$. Assign to the vertex $u$ of $G$ the color $\rho(S_u)$. Take two adjacent vertices $u$ and $v$ of $G$. The edge $e^{uv}$ in $G'$ is an edge between two vertices, one of $S_u$ and one of $S_v$: w.l.o.g. say $e^{uv} = \alpha^u\alpha^v$. This implies that $a_g^u\alpha^u$ and $a_g^v\alpha^v$ receive different colors and thus $\rho(S_u) \neq \rho(S_v)$. Hence this coloring of $G$ is a proper 3-vertex-coloring.

Conversely, suppose that $G$ admits a proper 3-vertex-coloring. Let $\rho$ be a partial edge-coloring of $G'$ with no colored edges. We choose the color $\rho(S_u)$ to be the color of $u$ in $G$ (and we color the appropriate edges of $G'$). By Claim 9, we can extend $\rho$ to each gadget $S_u$. Note that by the choice of $\rho(S_u)$, there is no conflict between edges of $S_u$ and $S_v$ when $u$ and $v$ are adjacent in $G$. It is left to color the edges of the form $e^{uv}$. By construction, there are only two edges at distance 2 of $e^{uv}$ (and this edge does not belong to a triangle). Hence there is at least one remaining color for $e^{uv}$. After coloring these edges, $\rho$ is an injective 3-edge-coloring of $G'$. □

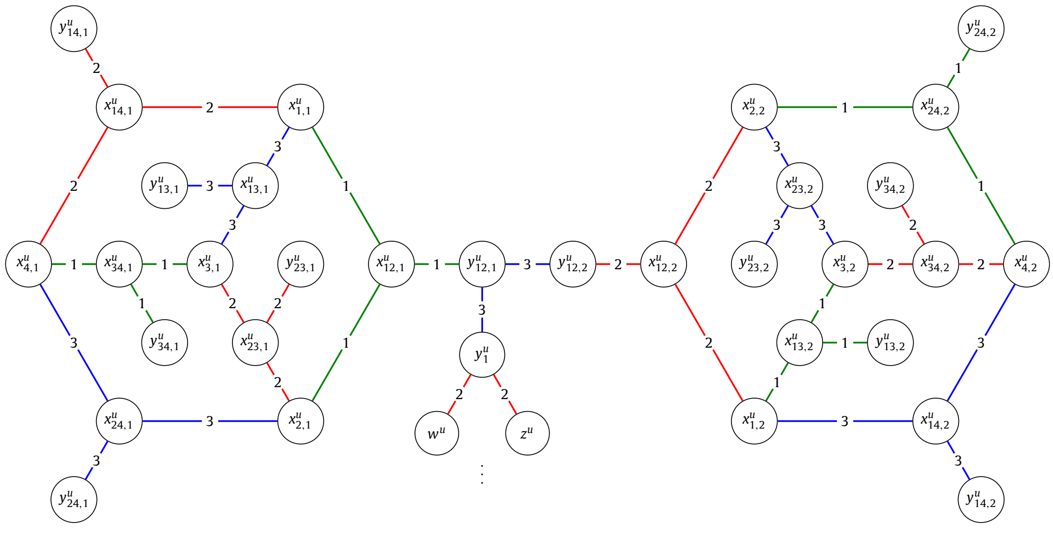



**Fig. 4.** Vertex gadget for planar bipartite subcubic graphs with girth at least 6.

### 3.2. Proof of Theorem 2.2

**Proof.** In order to prove this result, we will modify the previous construction to make it bipartite (the girth condition will be lost).

First we modify $S_u$ (see Fig. 4). Create the following gadget $H$. Start with a complete graph on four vertices $x_1, \dots, x_4$. For each edge $x_i x_j$, create a vertex $x_{ij}$ adjacent to both $x_i$ and $x_j$ and remove the edge $x_i x_j$. To each of these vertices of degree 2, add a pendant edge, with $y_{ij}$ the vertex of degree 1 adjacent to $x_{ij}$.

We claim that in every injective 3-edge-coloring $\gamma$ of $H$, for any $i \neq j$, the vertex $x_{ij}$ is incident to only one color. Suppose it is not the case, then there must exist an injective 3-edge-coloring $\gamma$ for which we have one of $x_{12}x_2$ and $x_{12}x_1$ colored differently from $x_{12}y_{12}$, say w.l.o.g. $\gamma(x_{12}x_1) = 1$ and $\gamma(x_{12}y_{12}) = 2$. We deduce that $\gamma(x_2x_{23}) = \gamma(x_2x_{24}) = 3$, $\gamma(x_{14}x_4) = \gamma(x_3x_{13}) = 2$, $\gamma(x_3x_{34}) = 1$, and there is no color available for $x_{23}y_{23}$, a contradiction.

Now, take two disjoint copies of $H$ named $H^u_1$ and $H^u_2$. Add an edge between the two vertices $y^u_{12,1}$ and $y^u_{12,2}$ and add the edge $y^u_{12,1}y^u_1$ where $y^u_1$ is a new vertex. Now repeat the construction process of $S_u$, for $g = 6$ for example, as described in the previous section by starting at the step where the vertices $w^u$ and $z^u$ are added. As we observed, the edges incident to vertex $x^u_{12,1}$ of $H^u_1$ (resp. $x^u_{12,2}$ of $H^u_2$) have the same color in any injective 3-edge-coloring $\rho$. Hence, $\rho(y^u_{12,1}y^u_{12,2}) = \rho(y^u_{12,1}y^u_1) \neq \rho(x^u_{12,1}y^u_{12,1})$. Note that this graph also admits an injective 3-edge-coloring (see Fig. 4). We are in the same configuration as in the proof of Theorem 2.1. Thus Claim 9 also holds for this gadget $S_u$. Note that this gadget is bipartite.

The edge gadget does not change, it is still the edge $e^{uv}$. We need to be careful with the bipartiteness of the constructed graph. To ensure that the constructed graph is bipartite, it suffices that all vertices $y^u_1$, $u \in V(G)$, belong to the same part of the bipartition. To that end, if there is a path of odd length between $y^u_1$ and $y^v_1$, then w.l.o.g. this path is $y^u_1 a^u_1 \dots a^u_g \alpha^u \alpha^v a^v_g \dots a^v_1 y^v_1$. If we increase the length of a sequence $a^u_1 \dots a^u_g$ in $S_u$ by 3 (and also adding $a'^u_g$, $a'^u_{g+1}$ and $a'^u_{g+2}$), then this path now has even length. With this trick, we can ensure the bipartiteness of the constructed graph $G'$ as well as keeping Claim 9 true in this new setting.

Hence, as before, $G$ admits a proper vertex-3-coloring if and only if $G'$ admits an injective 3-edge-coloring. □

## 4. Proof of Theorem 3

**Proof.** Let $G$ be a planar bipartite subcubic graph with girth at least 16. Let $A$ and $B$ be the two parts of the bipartition of $G$. We construct the graph $G_A$ as follows: for each $u \in A$, we create a vertex $u$ in $G_A$. For each pair of vertices $u$, $v$ of $A$ which are at distance 2, we add an edge between $u$ and $v$ in $G_A$. As $G$ is subcubic, a planar embedding of $G$ also serves as a planar embedding of $G_A$, where the edges of $G_A$ follow their corresponding path of length 2 in $G$. Hence, $G_A$ is a planar graph with maximum degree at most 6. Note that, by the girth condition on $G$, $G_A$ does not have any $k$-cycle, for all $k$ with $4 \leq k \leq 7$. Then, by the main result from [2], the graph $G_A$ admits a vertex-3-coloring $\gamma$.

We now color $G$ as follows: each edge $uv$ of $G$, where $u \in A$ and $v \in B$, is colored by the color $\gamma(u)$ in $G_A$. We claim that this is an injective 3-edge-coloring of $G$. Indeed, take any path $uvwz$ of $G$. W.l.o.g., assume $u, w \in A$ and $v, z \in B$. By construction, $uw \in E(G_A)$ and thus $uv$ and $wz$ receive different colors. □

## 5. Proof of Theorem 4

**Proof.** We give an fixed-parameter tractable (FPT) algorithm parameterized by the treewidth $tw(G)$ of our input graph $G$. We use a *nice tree decomposition* (see [13]) of the input graph for our dynamic programming algorithm. Nice tree decompositions are a well-known tool for designing algorithms on graphs of bounded treewidth using dynamic programming. In our notation, the set of vertices of the graph associated to a node $v$ of the tree, its *bag*, is denoted $X_v$.

A nice tree decomposition of a graph is a tree decomposition, rooted at a node $Root$, with the following types of nodes. A *join node* has exactly two children, with the same bags as their parent join node. An *introduce node* has a unique child and contains exactly one more vertex in its bag than its child's bag. A *forget node* also has a unique child, but the forget node's bag has exactly one less vertex than its child's bag. A *leaf node* is a leaf of the tree and contains no vertices. We call $G_{\leq v}$ the subgraph of $G$ induced by the subtree of the decomposition rooted at $v$ and $G_v$ the subgraph of $G$ induced by $X_v$. We note $N_H(u)$ for the neighborhood of a vertex $u$ in a subgraph $H$ of $G$.

We define the following set associated with a node $v$:

$$\mathcal{T}_v = \left\{t_1 : X_v \to \mathcal{P}(\{1,2,\dots,k\})^2\right\} \times \{t_2 : E(G_v) \to \{1,2,\dots,k\}\},$$

where $\mathcal{P}(X)$ is the power set of $X$. For $T \in \mathcal{T}_v$ with $T = (t_1, t_2)$, to simplify notation, we note $T[u]$ for $t_1(u)$ when $u \in X_v$ and $T[e] = t_2(e)$ when $e \in E(G_v)$. For a vertex $u \in X_v$, we also note $A_u$ and $B_u$ the two sets such that $T[u] = t_1(u) = (A_u, B_u)$.

The set $\mathcal{V}_{al}(v)$ is the subset of $\mathcal{T}_v$ such that $T \in \mathcal{V}_{al}(v)$ if and only if there exists an injective $k$-edge-coloring $\gamma$ of $G_{\leq v}$ such that:

1. for all $u \in X_v$, $A_u = \left\{\gamma(uw), w \in V(G_{\leq v}) \setminus X_v\right\}$, *i.e.* $A_u$ is the set of colors of the edges of $G_{\leq v}$ (not in $G_v$) incident with $u$,
2. for all $u \in X_v$, $B_u = \left\{\gamma(zw), zw \in E(G_{\leq v}) \setminus E(G_v) \text{ and } z \in N_{G_{\leq v}}(u)\right\}$, *i.e.* $B_u$ is the set of colors of the edges of $G_{\leq v}$ (not in $G_v$) at distance 2 of $u$ (or contained in a triangle containing $u$),
3. for all $e \in E(G_v)$, $T[e]$ is the color $\gamma(e)$.

In this case we say that $\gamma$ is *associated with* $T$. Note that for each injective $k$-edge-coloring of $G_{\leq v}$, there exists an associated $T \in \mathcal{T}_v$ and hence, $T \in \mathcal{V}_{al}(v)$. The set $\mathcal{V}_{al}(v)$ is thus the set of $T \in \mathcal{T}_v$ associated with an injective $k$-edge-coloring of $G_{\leq v}$.

Note that $\mathcal{V}_{al}(Root) \neq \emptyset$ if and only if there exists an injective $k$-edge-coloring of $G$. We will compute $\mathcal{V}_{al}(Root)$ with a dynamic programing algorithm. Also note that $|\mathcal{T}_v| \leq 2^{O(k\cdot tw(G)^2)}$.

First suppose that $v$ is a leaf node. Then $\mathcal{V}_{al}(v) = \mathcal{T}_v = \{(\emptyset, \emptyset)\}$.

Suppose that $v$ is a forget node where $v'$ is its child node such that $X_v \cup \{a\} = X_{v'}$. Let $T \in \mathcal{T}_v$, $T \in \mathcal{V}_{al}(v)$ if and only if there exists an associated coloring $\gamma$ of $G_{\leq v}$. This coloring $\gamma$ is also a coloring of $G_{\leq v'}$ and thus is associated to a $T' \in \mathcal{V}_{al}(v')$. In this case, since $T$ and $T'$ share the same coloring $\gamma$, we have the following constraints on $T$ and $T'$:

- for all $e \in E(G_v)$, $T[e] = T'[e] = \gamma(e)$,
- for all $u \in X_v$ such that $au \in E(G_{v'})$, $A_u = A'_u \cup \{T[au]\}$ and $B_u = B'_u \cup \{T[aw], w \in X_v \cap N_G(a), w \neq u\}$ where $T[u] = (A_u, B_u)$ and $T'[u] = (A'_u, B'_u)$,
- for all $u \in X_v$ such that $au \notin E(G_{v'})$, $A_u = A'_u$ and $B_u = B'_u \cup \{T[aw], w \in X_v \cap N_G(u) \cap N_G(a)\}$ where $T[u] = (A_u, B_u)$ and $T'[u] = (A'_u, B'_u)$.

The last two constraints reflect the fact that $A_u$ and $B_u$ must be updated after the removal of $a$. The only new colors that can be added to these sets come from edges incident with $a$. There are multiple cases, depending on whether $u$ and $a$ are adjacent or not, determining which colors of edges need to be added to these sets.

Hence, for all $T \in \mathcal{V}_{al}(v)$, it suffices to check whether there exists a $T' \in \mathcal{V}_{al}(v')$ for which the previous conditions are verified. This can be done in time $2^{O(k\cdot tw(G)^2)}$, as $T$ is uniquely determined by $T'$ in the above constraints.

Suppose that $v$ is an introduce node where $v'$ is its child node such that $X_v = X_{v'} \cup \{a\}$. Let $T \in \mathcal{T}_v$, $T \in \mathcal{V}_{al}(v)$ if and only if there exists an associated coloring $\gamma$ of $G_{\leq v}$. This coloring $\gamma$ is also a coloring of $G_{\leq v'}$ and thus is associated to a $T' \in \mathcal{V}_{al}(v')$. In other words $T$ is associated to a coloring $\gamma$ obtained by extending a coloring $\gamma'$ associated to some $T' \in \mathcal{V}_{al}(v')$. Thus $T' \in \mathcal{V}_{al}(v')$, we have the following constraints on $T$ and $T'$, in order to ensure that $\gamma$ is the extension of $\gamma'$:

- for all $e \in E(G_{v'})$, $T[e] = T'[e]$,
- for all $u \in X_{v'}$, $T[u] = T'[u]$,
- for $T[a] = (A_a, B_a)$, $A_a = \emptyset$ and $B_a = \bigcup_{u \in X_v, ua \in E(G_v)} A_u$,
- the coloring of $X_v$ is an injective $k$-edge-coloring,
- for all $ua \in E(G_v)$, $T[ua] \notin B_u \cup \bigcup_{u' \in X_v, u' \neq u, u'a \in E(G_v)} A_{u'}$.

The first two constraints correspond to the fact that $\gamma$ is an extension of $\gamma'$. As $a$ is a new vertex, $A_a = \emptyset$ and the only colors in $B_a$ can be obtained by edges incident with some vertex $u \in X_v$ itself adjacent to $a$, hence the third constraint. The last two constraints correspond to the fact that the coloring of the new edges around $a$ cannot be in conflict with edges already colored. The fourth constraint checks that no such conflict arises in $X_v$ and the fifth constraint ensures that for each new edge $ua$ the color $T[ua]$ does not appear around an edge at distance 2 from $a$ or $u$. For each $T'$, there are at most $2^{tw(G)}$ possible candidates to be added to $\mathcal{V}_{al}(v)$. Hence $2^{O(k\cdot tw(G)^2)}$ time is sufficient to compute $\mathcal{V}_{al}(v)$ from $\mathcal{V}_{al}(v')$.

Suppose that $v$ is a join node where $v_1$ and $v_2$ are its children nodes such that $X_v = X_{v_1} = X_{v_2}$. Let $T \in \mathcal{T}_v$, $T \in \mathcal{V}_{al}(v)$ if and only if there exists an associated coloring $\gamma$ of $G_{\leq v}$. As both $G_{\leq v_1}$ and $G_{\leq v_2}$ are subgraphs of $G_{\leq v}$, $\gamma$ is also a coloring of $G_{\leq v_i}$ ($i \in \{1,2\}$) and thus is associated to a $T_i \in \mathcal{V}_{al}(v_i)$. In this case, since $T$, $T_1$ and $T_2$ share the same coloring $\gamma$, we have the following constraints on $T$, $T_1$ and $T_2$:

- for all $e \in E(G_v)$, $T[e] = T_1[e] = T_2[e]$,
- for all $u \in X_v$, $A_u = A^1_u \cup A^2_u$ and $B_u = B^1_u \cup B^2_u$ where $T_i[u] = (A^i_u, B^i_u)$ for $i \in \{1,2\}$,
- for all $uw \in E(G_v)$, $A_u \cap A_w = \emptyset$.

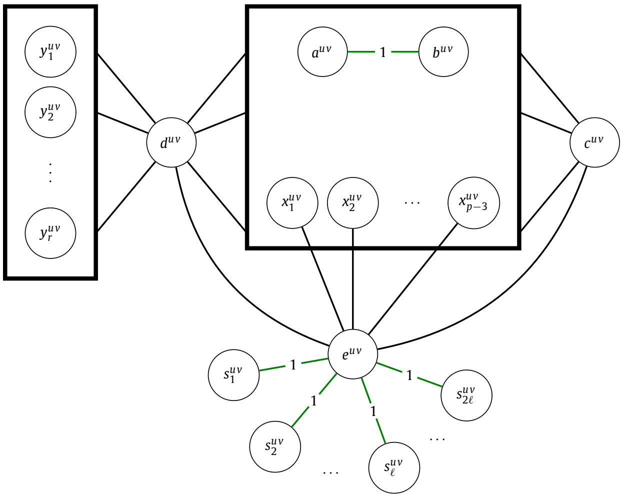



**Fig. 5.** The edge gadget $E_{uv}$ when $r > 0$. The vertices inside each of the two rectangle form a clique. The vertex $c^{uv}$ is adjacent to every vertex inside the largest rectangle. The vertex $d^{uv}$ is adjacent to every vertex inside the two rectangles.

The last constraint corresponds to the fact that the coloring is an injective $k$-edge-coloring (*i.e.* with no conflicts between the two subtrees). Given $T_1 \in \mathcal{V}_{al}(v_1)$ and $T_2 \in \mathcal{V}_{al}(v_2)$, $T$ is uniquely determined by the above constraints. Hence it suffices to try all the pairs of $T_1, T_2$ and when the obtained set $T$ verifies all conditions, we can add it to $\mathcal{V}_{al}(v)$. This can be done in time $(2^{O(k \cdot tw(G)^2)})^2 = 2^{O(k \cdot tw(G)^2)}$. $\square$

## 6. Proof of Theorem 5

**Proof.** We reduce from $k$-Edge-Coloring, proven to be NP-Complete even for $k$-regular graphs in [15].

$k$-Edge-Coloring
Instance: A $k$-regular graph $G$.
Question: Does $G$ admit a proper $k$-edge-coloring?

We choose $p$ to be the largest integer such that $k = \binom{p}{2} + r$ (and thus $r < p$) and recall that $k \geq 45$. Moreover we set $\ell = 2p$.

Let $G$ be the input $k$-regular graph. For $uv \in E(G)$, we define the edge gadget $E_{uv}$ as follows (see Fig. 5). First create the following vertices $a^{uv}$, $b^{uv}$, $x_1^{uv}$, ..., $x_{p-3}^{uv}$, $c^{uv}$, $d^{uv}$, $e^{uv}$, $y_1^{uv}$, ..., $y_r^{uv}$, $s_1^{uv}$, ..., $s_{2\ell}^{uv}$. The vertices $s_i^{uv}$ have degree 1 in $E_{uv}$ and will be connected to the rest of the graph. The vertices $\left\{x_1^{uv}, \dots, x_{p-3}^{uv}, a^{uv}, b^{uv}, c^{uv}\right\}$ form a clique; this is also the case for $\left\{x_1^{uv}, \dots, x_{p-3}^{uv}, a^{uv}, b^{uv}, d^{uv}\right\}$ and $\left\{y_1^{uv}, \dots, y_r^{uv}, d^{uv}\right\}$. The vertex $e^{uv}$ is adjacent to $c^{uv}$, $d^{uv}$, $x_1^{uv}$, ..., $x_{p-3}^{uv}$, $s_1^{uv}$, ..., $s_{2\ell}^{uv}$. In the case where $r = 0$, *i.e.* $k = \binom{p}{2}$, we delete $d^{uv}$.

Let $u$ be a vertex of $G$ with $v_1, \dots, v_k$ its neighbors. We construct the vertex gadget $S_u$ from $k \times \ell$ vertices $v_{1,1}, \dots, v_{1,\ell}, v_{2,1}, \dots, v_{k,\ell}$ and successively consider pairs $v_i$, $v_j$ of neighbors. For each pair, we add an edge between one of $v_{i,1}, \dots, v_{i,\ell}$ of minimum degree and one of $v_{j,1}, \dots, v_{j,\ell}$ with minimum degree. By adding edges one by one in this way, we ensure that the maximum degree of the vertices of $S_u$ is at most $\frac{k}{\ell} + 1$.

Finally, for each edge $uv$ of $G$, we identify the $2\ell$ vertices $s_1^{uv}$, ..., $s_{2\ell}^{uv}$ with the $\ell$ vertices of $S_u$ corresponding to $v$ (since $v$ is a neighbor of $u$, by the construction of $S_u$ in the previous paragraph, there are $\ell$ such vertices in $S_u$) and with the $\ell$ vertices of $S_v$ corresponding to $u$. This creates the graph $G'$. Note that its maximum degree is $\max(2\ell + p - 1, \frac{k}{\ell} + 2) \leq 5p \leq 5\sqrt{3k}$.

**Claim 10.** *For any injective $k$-edge-coloring $\gamma$ of $E_{uv}$, we have $\gamma(e^{uv}s_1^{uv}) = \gamma(e^{uv}s_2^{uv}) = \cdots = \gamma(e^{uv}s_{2\ell}^{uv})$. Moreover if $\gamma$ is a partial injective $k$-edge-coloring of $E_{uv}$ where $\gamma(e^{uv}s_1^{uv}) = \gamma(e^{uv}s_2^{uv}) = \cdots = \gamma(e^{uv}s_{2\ell}^{uv})$ and there are no other colored edges, we can extend $\gamma$ to $E_{uv}$.*

**Proof.** First note that the clique $\left\{x_1^{uv}, \dots, x_{p-3}^{uv}, a^{uv}, b^{uv}, c^{uv}\right\}$ needs exactly $\binom{p}{2}$ distinct colors. W.l.o.g. $a^{uv}b^{uv}$ is colored 1 and the colors used for this clique are 1, 2, ..., $\binom{p}{2}$. None of these colors can be used to color the $r$ edges of the form $d^{uv}y_i^{uv}$ hence they must be colored with $\binom{p}{2} + 1, \dots, \binom{p}{2} + r$. One can observe that an edge $e^{uv}s_i^{uv}$ cannot have a color among $\binom{p}{2} + 1, \dots, \binom{p}{2} + r$ as it is at distance 2 from the edges of the form $d^{uv}y_j^{uv}$ ($j \in \{1, \dots, r\}$). Moreover this edge cannot receive the same color as one of the edges of the clique $\left\{x_1^{uv}, \dots, x_{p-3}^{uv}, a^{uv}, b^{uv}, c^{uv}\right\}$ except for the color 1 on the edge $a^{uv}b^{uv}$. Hence all edges of the form $e^{uv}s_i^{uv}$ have the same color.

Now suppose we have a coloring $\gamma$ such that these edges $e^{uv}s_i^{uv}$ ($i \in \{1, \dots, 2\ell\}$) are all colored with the same color, say 1. We color $a^{uv}b^{uv}$ with color 1 and use the $\binom{p}{2} + r - 1$ other colors to color the rest of the edges of the clique $\left\{x_1^{uv}, \dots, x_{p-3}^{uv}, a^{uv}, b^{uv}, c^{uv}\right\}$ and the edges of the form $d^{uv}y_j^{uv}$ ($j \in \{1, \dots, r\}$). We color $e^{uv}z$ for $z \in \left\{x_1^{uv}, \dots, x_{p-3}^{uv}, c^{uv}\right\}$ with the color of $a^{uv}z$.

If $r = 0$, then $E_{uv}$ is colored and $\gamma$ is an injective $k$-edge-coloring.

If $r > 0$, we color $d^{uv}e^{uv}$ and $d^{uv}a^{uv}$ with the color of $d^{uv}y_1^{uv}$. We color $d^{uv}z$ for $z \in \left\{x_1^{uv}, \ldots, x_{p-3}^{uv}, b^{uv}\right\}$ with the color of $c^{uv}z$. It is left to color the edges of the clique $\left\{y_1^{uv}, \ldots, y_r^{uv}\right\}$, for which we have available the $\binom{p-1}{2}$ colors used to color the clique $\left\{x_1^{uv}, \ldots, x_{p-3}^{uv}, a^{uv}, b^{uv}\right\}$, which is enough as $r \leq p-1$. This is an injective $k$-edge coloring of $E_{uv}$. ■

Suppose there is an injective $k$-edge-coloring $\gamma$ of $G'$. For an edge $uv$ of $G$, we color it with the color $\gamma(e^{uv}s_1^{uv})$. Take two adjacent edges of $G$: $uv_1$ and $uv_2$. In $S_u$, there is an edge between $v_{1,i}$ and $v_{2,j}$ for some indices $i$ and $j$. Thus the edges $e^{uv_1}v_{1,i}$ and $e^{uv_2}v_{2,j}$ receive different colors. By Claim 10, $uv_1$ and $uv_2$ receive different colors. Hence $G$ admits a $k$-edge-coloring.

Suppose there is a $k$-edge coloring $\gamma$ of $G$. For each edge $uv$, we color $e^{uv}s_i^{uv}$ with the color $\gamma(uv)$. By Claim 10, we can extend this coloring to all $E_{uv}$. At this point there is no conflict between the colored edges. Indeed the only pairs of edges which are at distance 2 and not in the same edge gadget are of the form $e^{uw}s_i^{uw}$, and since $\gamma$ is proper, there is no conflict here. It is left to color the edges inside the vertex gadget. Let $e = v_{i,j}v_{i',j'}$ be an uncolored edge. As the maximum degree of the vertices of $S_u$ is at most $\frac{k}{\ell}+2$, there are at most $(\frac{k}{\ell}+2)^2$ edges incident to a vertex of $S_u$ that can be in conflict with $e$. We must also consider the edges incident with $e^{uv_i}$ and $e^{uv_j}$. For each of the two vertices there is one forbidden color $\gamma(uv_i)$ which is common to $2\ell$ edges incident to $e^{uv_i}$ to which we need to add $p-1$ colors for the other edges of $e^{uv_i}$. In the end, there are at most $2p + (\frac{k}{\ell}+2)^2$ forbidden colors for $e$. As $2p + (\frac{k}{\ell}+2)^2 \leq 2p + (\frac{p-1}{4}+2)^2 = (\frac{p-1}{4})^2 + 3p + 3 \leq k$ when $k \geq 45$ and $p \geq 10$, $G'$ admits an injective $k$-edge-coloring. □

## 7. Conclusion

We proved that Injective 3-Edge-Coloring and Injective 4-Edge-Coloring are NP-complete on some restricted classes of subcubic graphs. One can ask whether Injective 5-Edge-Coloring is NP-complete on subcubic graphs. A conjecture proposed by Ferdjallah *et al.* [7] states that every subcubic graph admits an injective 6-edge-coloring (it is proved for planar graphs in [14]). In fact, we only know of two connected subcubic graphs which require six colors: $K_4$ and the prism. Perhaps these are the only examples that are not 5-colorable, in which case Injective 5-Edge-Coloring would be polynomial-time solvable for this class.

We have also proved that for planar bipartite subcubic graphs, Injective 3-Edge-Coloring is polynomial-time solvable when the girth is at least 16 (because the answer is always YES), but NP-Complete when the girth is 6. It would be interesting to determine the values of the girth of planar bipartite subcubic graphs for which Injective 3-Edge-Coloring stays NP-Complete, becomes polynomial-time solvable, and always has YES as an answer.

We also do not know whether Injective 4-Edge-Coloring is NP-Complete for bipartite subcubic graphs.

## 8. Acknowledgments

This research was supported by the IFCAM project "Applications of graph homomorphisms" (MA/IFCAM/18/39) and by the ANR project HOSIGRA (ANR-17-CE40-0022).

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

[1] Maria Axenovich, Philip Dörr, Jonathan Rollin, Torsten Ueckerdt, Induced and weak induced arboricities, Discrete Math. 342 (2) (2019) 511–519.
[2] Oleg V. Borodin, Aleksey N. Glebov, André Raspaud, Mohammad R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, J. Comb. Theory, Ser. B 93 (2) (2005) 303–311.
[3] Yuehua Bu, Chentao Qi, Injective edge coloring of sparse graphs, Discrete Math. Algorithms Appl. 10 (02) (2018) 1850022.
[4] Domingos M. Cardoso, J. Orestes Cerdeira, J. Pedro Cruz, Charles Dominic, Injective edge coloring of graphs, Filomat 33 (2019) 6411–6423.
[5] Richard Cole, Lukasz Kowalik, New linear-time algorithms for edge-coloring planar graphs, Algorithmica 50 (3) (2008) 351–368.
[6] Paul Erdős, Problems and results in combinatorial analysis and graph theory, in: J. Akiyama, Y. Egawa, H. Enomoto (Eds.), Graph Theory and Applications, in: Annals of Discrete Mathematics, vol. 38, Elsevier, 1988, pp. 81–92.
[7] Baya Ferdjallah, Samia Kerdjoudj, André Raspaud, Injective edge-coloring of sparse graphs, arXiv e-print: arXiv:1907.09838, 2019, Manuscript.
[8] Jean-Luc Fouquet, Jean-Loup Jolivet, Strong edge-coloring of graphs and applications to multi-*k*-gons, Ars Comb. 16A (1983) 141–150.
[9] Michael R. Garey, David S. Johnson, Larry Stockmeyer, Some simplified NP-complete graph problems, Theor. Comput. Sci. 1 (3) (1976) 237–267.
[10] Geňa Hahn, Jan Kratochvíl, Josef Širáň, Dominique Sotteau, On the injective chromatic number of graphs, Discrete Math. 256 (1) (2002) 179–192.
[11] Hervé Hocquard, Pascal Ochem, Petru Valicov, Strong edge-colouring and induced matchings, Inf. Process. Lett. 113 (19) (2013) 836–843.
[12] Ian Holyer, The NP-completeness of edge-coloring, SIAM J. Comput. 10 (4) (1981) 718–720.
[13] Ton Kloks, Treewidth, Computations and Approximations, Springer, 1994.
[14] Alexandr V. Kostochka, André Raspaud, Jingwei Xu, Injective edge coloring of graphs with given maximum degree, arXiv e-print: arXiv:2010.00429, 2020, Manuscript.
[15] Daniel Leven, Zvi Galil, NP completeness of finding the chromatic index of regular graphs, J. Algorithms 4 (1) (1983) 35–44.
[16] Jun Yue, Shiliang Zhang, Xia Zhang, Note on the perfect EIC-graphs, Appl. Math. Comput. 289 (2016) 481–485.