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BOUNDING THE ORDER OF A GRAPH USING ITS DIAMETER AND METRIC DIMENSION: A STUDY THROUGH TREE DECOMPOSITIONS AND VC DIMENSION*

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Abstract. The metric dimension of a graph is the minimum size of a set of vertices such that each vertex is uniquely determined by the distances to the vertices of that set. Our aim is to upper-bound the order n of a graph in terms of its diameter d and metric dimension k. In general, the bound $n \leq d^k + k$ is known to hold. We prove a bound of the form $n = \mathcal{O}(kd^2)$ for trees and outerplanar graphs (for trees we determine the best possible bound and the corresponding extremal examples). More generally, for graphs having a tree decomposition of width w and length ℓ, we obtain a bound of the form $n = \mathcal{O}(kd^2(2\ell+1)^{3w+1})$. This implies in particular that $n = \mathcal{O}(kd^{\mathcal{O}(1)})$ for graphs of constant treewidth and $n = \mathcal{O}(f(k)d^2)$ for chordal graphs, where f is a doubly exponential function. Using the notion of distance-VC dimension (introduced in 2014 by Bousquet and Thomassé) as a tool, we prove the bounds $n \leq (dk+1)^{t-1}+1$ for K_t-minor-free graphs and $n \leq (dk+1)^{d(3\cdot 2^r+2)}+1$ for graphs of rankwidth at most r.

Key words. graph theory, VC dimension, metric dimension

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1. Introduction.

A *resolving set* of a graph is a set of vertices that uniquely determines each vertex by means of the ordered set of distances to the vertices in the resolving set. The *metric dimension* of the graph is the smallest size of a resolving set. These concepts, introduced independently by Slater [29] (who called resolving sets *locating sets*) and by Harary and Melter [19], have been widely studied since then; see, for example, the papers [2, 5, 12, 20, 21, 26]. More generally, they fit into the topic of *identification* or *separation* problems in discrete structures, such as separating systems, distinguishing sets, and related concepts (for a few references, see [7, 8, 11, 22]). These concepts have many applications and connections to other areas. For example, the metric dimension can be applied to network discovery [4, 3], robot navigation [21], coin-weighing problems [26], T-joins [26], the Mastermind game [13], or chemistry [12].

The goal of this paper is to study the relation between the order, the diameter, and the metric dimension of graphs, in particular for graphs belonging to specific graph classes.

Important concepts and definitions. All considered graphs are finite and simple. We will denote by $N[v]$ the *closed neighborhood* of vertex v and by $N(v)$ its

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open neighborhood $N[v] \setminus \{v\}$. Let $d_G(u,v)$, or simply $d(u,v)$ if there is no ambiguity, denote the distance between two vertices u and v in graph G. Similarly, for two sets X and Y of vertices of G, $d_G(X,Y)$ denotes the shortest distance between a vertex of X and a vertex of Y.

DEFINITION 1. *A set R of vertices of a graph G is a* resolving set *if for each pair u, v of distinct vertices, there is a vertex x of R with $d(x,u) \neq d(x,v)$. The smallest size of a resolving set of G is the* metric dimension *of G.*

A graph is said to be *chordal* if it has no induced cycle of length at least 4. A graph is *planar* if it has an embedding in the plane that induces no edge-crossing. It is outerplanar if it is planar and has an embedding in the plane where each vertex lies on the outer face. A *minor* of a graph is a graph obtained by a succession of vertex- and edge-deletions and edge-contractions. We say that a graph G is H-minor-free if H is not a minor of G. By the Graph Minor Theorem [24], any minor-closed class of graphs (such as the classes of planar graphs, outerplanar graphs, or graphs with treewidth at most w) is defined by a finite set of forbidden minors.

Previous work. One can easily observe that in a graph G of diameter d and with metric dimension k and n vertices, we have the bound $n \leq d^k + k$ [12, 21]. Indeed, given a resolving set R of size k, every vertex outside of R can be associated to a distinct vector of length k and values ranging from 1 to d. This trivial bound, however, is only tight for $d \leq 3$ or $k = 1$ [20]. Nevertheless, the more precise (and tight) bound $n \leq (\lfloor 2d/3 \rfloor + 1)^k + k\sum_{i=1}^{\lceil d/3 \rceil}(2i-1)^{k-1}$ is given in [20]. It is natural to ask for which kind of graphs a bound of this form is tight. We therefore wish to study the following problem.

PROBLEM 1. *Given a graph class $\mathcal{C}$, determine the largest possible order of a graph in $\mathcal{C}$ having metric dimension k and diameter d.*

This problem was considered by the third and fifth authors, together with Mertzios, Naserasr, and Valicov [17]. These authors studied interval graphs and permutation graphs and proved bounds of the form $n = \mathcal{O}(dk^2)$. These bounds were shown to be best possible (up to constant factors). In the case of unit interval graphs, bipartite permutation graphs, and cographs, it was proved in the same paper that $n = \mathcal{O}(dk)$.

Surprisingly, the above problem seems to have not been studied even for trees, despite the fact that the metric dimension of trees is well understood (see [12, 21, 29]). In this paper, we answer this question. We extend our result for trees in two ways. First, we give bounds involving the length and width of a tree decomposition of the graph. Second, we study graphs that have bounded *distance-VC dimension.* (These notions will be defined in the corresponding sections of the paper.)

As further recent work related to this paper, we remark that the metric dimension of t-trees has recently been investigated in [5], and the treelength of a graph has recently been used to design algorithms to compute the metric dimension [6]. Algorithms and complexity results regarding the computation of the metric dimension of graphs belonging to graph classes considered in the present paper can be found in [14, 16, 18].

Our results and structure of the paper. In the first part of the paper, section 2, we study trees and generalize our method using the tool of tree decompositions. We start in section 2.1 by an exact bound of the form $n = (\frac{1}{8} + o(1))d^2k$ for trees of order n, metric dimension k, and diameter d, and we characterize the trees reaching our bound. We then show in section 2.2 that a graph with a tree decomposition of width w and length ℓ satisfies $n = \mathcal{O}(kd^2(2\ell+1)^{3w+1})$. This implies the bound $n = \mathcal{O}(k2^{3w}d^{3w+3})$ for graphs of treewidth at most w and $n = \mathcal{O}(kd^2 3^{3\omega-2})$ for chordal graphs with maximum clique ω.

The second part of the paper, section 3, is devoted to the use of the distance-VC dimension. We first show (using the notion of test covers) how the VC dimension of the ball hypergraph of a graph can be used to derive a general bound on the order using the diameter and the metric dimension. We then bound the dual distance-VC dimension of K_t-minor-free graphs and graphs of rankwidth at most r, which implies the bounds $n \leq (dk+1)^{t-1}+1$ and $n \leq (dk+1)^{d(3\cdot 2^r+2)}+1$, respectively. In particular, this shows that for planar graphs, we have $n \leq (dk+1)^4$; this partially answers an open question from [17]. We then use a completely different method in section 3.4 to prove that $n = \mathcal{O}(kd^2)$ for outerplanar graphs, which we show to be tight.

Finally, we conclude in section 4 with some open questions.

2. Trees and graphs with specific tree decompositions. We first study Problem 1 for graphs admitting specific types of tree decompositions. We start with trees, which form the class of nontrivial graphs that is the simplest (with respect to tree decompositions).

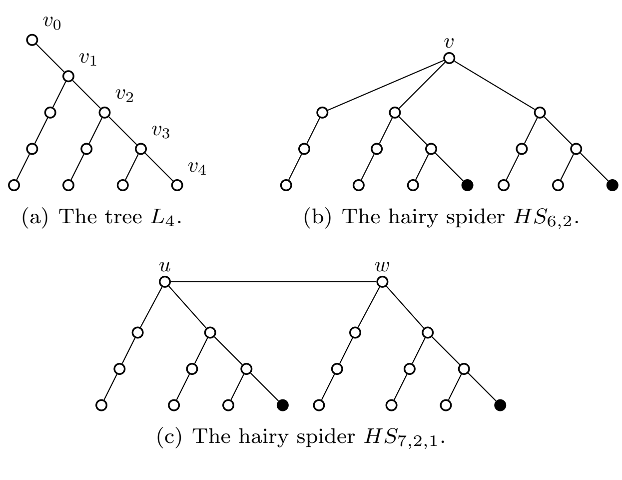
2.1. Trees. We first give the constructions of some extremal trees. See Figure 1 for illustrations.

For $r \in \mathbb{N}$, let L_r be the rooted tree obtained from a path $v_0, v_1, \ldots, v_r$ rooted at v_0 by attaching a path of length $r-i$ to vertex v_i for $i = 1, 2, \ldots, r-1$. Denote a path of length r rooted at one of its end vertices by P_r^*. Let $k \in \mathbb{N}$ with $k \geq 2$.

For even $d \in \mathbb{N}$, we define the *hairy spider* $HS_{d,k}$ as the tree obtained from k disjoint copies of $L_{d/2}$ and a path $P_{d/2}^*$ by identifying their roots to a vertex v. For odd $d \in \mathbb{N}$ with $d \geq 3$ and for $a \in \mathbb{N}$ with $0 \leq a \leq k$, we define $HS_{d,k,a}$ as the tree obtained from k disjoint copies of $L_{(d-1)/2}$, two copies of $P_{(d-1)/2}^*$, and a path on two vertices, u and w, by identifying the roots of a copies of $L_{(d-1)/2}$ and of one copy of $P_{(d-1)/2}^*$ with u and the roots of the remaining $k-a$ copies of $L_{(d-1)/2}$ and the root of the other copy of $P_{(d-1)/2}^*$ with w.

THEOREM 2. *Let T be a tree of diameter d and metric dimension k, where $k \geq 2$. Then*

$$|V(T)| \leq \begin{cases} \frac{1}{8}(kd+4)(d+2) & \text{if } d \text{ is even,} \\ \frac{1}{8}(kd-k+8)(d+1) & \text{if } d \text{ is odd.} \end{cases}$$



(a) The tree L_4. (b) The hairy spider $HS_{6,2}$.

(c) The hairy spider $HS_{7,2,1}$.

FIG. 1. *Extremal trees. Black vertices form optimal resolving sets.*

Equality holds for even d if and only if $T = HS_{d,k}$ and for odd d if and only if $T = HS_{d,k,a}$ for some integer a with $0 < a < k$.

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a resolving set for T. Let C be the set of *central vertices* of T, that is, the set of vertices of T that minimize the maximum distance to all the other vertices of the tree. For $i = 1, 2, \ldots, k$, let P_i be the shortest path from C to x_i. Let $r = \max_{v \in V(T)} d_T(v, C)$, so if d is even, then $r = d/2$ and r is the radius of T, and if d is odd, then $r = (d-1)/2$ since for odd d every vertex is within distance $(d-1)/2$ of the nearest central vertex of T. Define the subtree T_S of T by

$$T_S = T[C] \cup \bigcup_{i=1}^{k} P_i. \tag{1}$$

For $v \in V(T_S)$, we define T_v to be the largest subtree of T containing v and no other vertex of T_S. In other words, T_v is the union of all branches of T at v not containing any edge of T_S. Possibly, $T_v = K_1$. We first show that for every vertex v of T_S,

$$T_v \text{ is a path with } v \text{ as an end-vertex.} \tag{2}$$

We first show that v is an end-vertex (that is, a leaf) of T_v. Indeed, if v had two neighbors in T_v, then they would have the same distance to every vertex in S, and so S would not resolve them, a contradiction. The same argument shows that no vertex of T_v has degree greater than two. This shows (2).

Hence, T is obtained from T_S by appending a path on $|V(T_v)| - 1$ vertices to v for all $v \in V(T_S)$. We now bound the length of this path by showing that

$$|V(T_v)| \le r - d_T(v, C) + 1. \tag{3}$$

Let v' be the end-vertex of T_v with $v' \ne v$. Then $d_T(v', v) = |V(T_v)| - 1$. Hence,

$$r \ge d_T(v', C) = d_T(v', v) + d_T(v, C) = |V(T_v)| - 1 + d_T(v, C),$$

and (3) follows.

From (1), we obtain

$$\begin{aligned} n &= \sum_{v \in V(T_S)} |V(T_v)| \\ &\le \sum_{v \in C} |V(T_v)| + \sum_{i=1}^{k} \sum_{w \in V(P_i) - C} |V(T_w)|. \end{aligned} \tag{4}$$

By (3), we have $|V(T_v)| \le r + 1$ for all $v \in C$. The vertices of P_i are at distance $0, 1, \ldots, \ell_i$ from C in T, where ℓ_i is the length of P_i. Hence, by (3) and $\ell_i \le r$, we get

$$\sum_{w \in V(P_i) - C} |V(T_w)| \le \sum_{j=1}^{\ell_i} (r - j + 1) \le \sum_{j=1}^{r} (r - j + 1) = \frac{1}{2} r(r+1). \tag{5}$$

In total, we obtain

$$n \le |C|(r+1) + \frac{k}{2} r(r+1). \tag{6}$$

If d is even, then $r = \frac{d}{2}$ and C contains only one vertex. Hence,

$$n \leq (r+1) + \frac{k}{2}r(r+1) = \frac{kr+2}{2}(r+1) = \frac{(kd+4)(d+2)}{8},$$

and the desired bound follows in this case. If d is odd, then $r = \frac{d-1}{2}$ and C contains exactly two vertices. Hence,

$$n \leq 2(r+1) + \frac{k}{2}r(r+1) = \frac{kr+4}{2}(r+1) = \frac{(kd-k+8)(d+1)}{8},$$

and the desired bound follows also in this case.

Now assume that T is a tree of diameter d and metric dimension k attaining the bound. Then equality holds also in (6) in (4)–(5). So the paths P_i share no vertices other than central vertices, and each path has length r. Moreover, equality in (5) implies that for the vertices v of P_i, the trees T_v have order $2, 3, \dots, r-1$, respectively, for each i. Equality in (6) implies also that for each central vertex v, the tree T_v has $r+1$ vertices. In total, it follows that $T = HS_{d,k}$ if d is even and that, if d is odd, $T = HS_{d,k,a}$ for some $a \in \{0, 1, \dots, k\}$. It is easy to see that the trees $HS_{d,k,0}$ and $HS_{d,k,k}$ have metric dimension $k+1$, so we conclude that $T = HS_{d,k,a}$ for some $a \in \{1, 2, \dots, k-1\}$. ∎

2.2. Using tree decompositions. We now generalize our result for trees to graphs with tree decompositions of given width and length. These results also generalize results of [17] for interval graphs and permutation graphs (which have treelength at most 1 and 2, respectively [6]).

We first recall the definition of *tree decomposition* introduced by Robertson and Seymour [23]. We shall copy the definition given by Dourisboure and Gavoille [15], which is slightly lighter in terms of indices.

Definition 3 (tree decomposition [15]). *Let G be a graph. A* tree decomposition *of G is a tree T whose vertices, called* bags*, are subsets of $V(G)$ such that the following properties are satisfied:*

(P1) $\bigcup_{X \in V(T)} X = V(G)$;
(P2) *for every edge e of G, there exists X in $V(T)$ such that both ends of e are in X;*
(P3) *for $X, Y,$ and Z in $V(T)$, if Y lies on the path in T from X to Z, then $X \cap Z \subseteq Y$.*

As mentioned in [15], property (P3) of Definition 3 implies that, for any vertex x in $V(G)$, the set of bags containing x induces a subtree of T. The classic *width* parameter of a tree decomposition is defined as $\max\{|X| - 1 : X \in V(T)\}$. For any bag X in $V(T)$, the *diameter* of X is the maximum distance $d_G(x, y)$ over every pair of vertices x and y in X. (Note that here the distance is taken in G and not in $G[X]$.) The *length* of a tree decomposition is the largest diameter of a bag over every bag X in $V(T)$ [15]. The *treewidth* (resp., *treelength*) of a graph G is the minimum width (resp., length) among all tree decompositions of G.

A tree decomposition is *reduced* if no bag is a subset of another bag. One may easily check that any tree decomposition can be turned into a reduced tree decomposition by removing the bags which are not maximal with respect to inclusion and without altering the width and the length of the decomposition.

A *cutset* of a graph G is a set of vertices in G whose removal increases the number of components. We shall prove the following theorem.

THEOREM 4. *Let G be a graph of order n and diameter d. Let T be a reduced tree decomposition of G of length ℓ and width w. If there is a resolving set of size k in G, then*

$$n = \mathcal{O}(kd^2(2\ell+1)^{3w+1}).$$

Proof. Let G be a graph of diameter d with a resolving set S of size k. Let T be a tree decomposition of G with length ℓ and width w. The following claim is easily derived from the definition of a tree decomposition.

CLAIM 4.A. *Every bag X which is not a leaf in T is a cutset for G.*

For an easier reading of the following proofs, let us pick an arbitrary root X_r for T. For any bag X in $V(T)$, we define the subtree $T(X)$ as the subtree of T induced by X and all its descendants.

CLAIM 4.B. *Let X be a bag in $V(T)$ such that for every bag Y in $T(X)$, the set $Y \cap S$ is included in X. Let A be the set defined as*

$$A = \bigcup_{Y \in V(T(X))} Y.$$

Then

$$|A| \leq (d+1)(2\ell+1)^w.$$

Proof of claim. For any vertex x in A, every path from x to an element of S has to go through X (this is implied by Claim 4.A). Therefore, the distances from x to the vertices of S are completely determined by the distances from x to the vertices of X. Since S is a resolving set, the vertices in A must all have a different distance vector to X. By taking a specific vertex of X as a pin point, the distance from x to this pin is at most d, and all other distances can only differ from this distance by at most ℓ. There are at most w other vertices in X. Thus, the number of possible vectors is smaller than or equal to

$$(d+1)(2\ell+1)^w,$$

concluding the proof of Claim 4.B. ∎

For each vertex v, we call the bag in T that contains v and is at minimum distance from the root X_r of T the *oldest bag in T containing v*. We note that such a bag is uniquely defined because of the subtree structure and the properties of a tree decomposition T. For every vertex s in the resolving set S, we denote the oldest bag in T containing s by X_s, and we call it the *ancestor* of s in T.

Let T_S be the subtree of T obtained by only considering the ancestors of all s in S and the paths from them to the root X_r. Any leaf of T_S is the ancestor of some s in S. As a direct consequence, T_S has at most k leaves. A *thread* in a graph G is a path all whose inner vertices have degree 2 in G.

CLAIM 4.C. *Let P be a thread of length L in T_S. Let X_0 and X_l be the bags at both ends of P. Suppose that for every inner vertex X of P, the set $X \cap S$ is included in $X_0 \cup X_L$. Then*

$$L \leq (\ell+1)(2\ell+1)^{2w+1}\left[d_G(X_0, X_l)+1\right].$$

Proof of claim. Let λ be the distance in G between the sets X_0 and X_L. Let $x_0x_1 \cdots x_\lambda$ be a shortest path in G between X_0 and X_L (x_0 is in X_0, and x_λ is in X_L). Note that every edge along the path $x_0x_1 \cdots x_\lambda$ must be in one of the bags along P.

If x is in X_i and X_{i+t} for some i, j along the thread, then it is in all the bags in between X_i and X_{i+t} (by the connectivity condition). Notice that every path between a vertex in $\bigcup_{z=i}^{i+t} X_z$ and a vertex in S has to go through X_i or X_{i+t}. This means that all vertices in $\bigcup_{z=i}^{i+t} X_z$ must have different distance vectors to $X_i \cup X_{i+t}$.

These distances are bounded above by 2ℓ since x is in all the bags along this thread. The distance to x is at most ℓ. There are at most $2w$ vertices different from x in $X_i \cup X_{i+t}$. We may conclude that

$$\left|\bigcup_{z=i}^{i+t} X_z\right| \leq (\ell+1)(2\ell+1)^{2w}.$$

Since the tree decomposition is reduced, every bag X_i must contain a vertex which is not in any X_j for j between 0 and $i-1$. We derive that the number of bags is smaller than the number of vertices:

$$k+1 \leq (\ell+1)(2\ell+1)^{2w}. \tag{7}$$

In other words, vertices cannot be in too many bags along the thread.

Now, we shall prove that L cannot be too big with respect to λ. For this, let us denote by i_q the largest index of a bag containing x_q for q between 0 and λ:

$$i_q = \max\{i : x_q \in X_i\}.$$

With the help of (7), we may say that

$$i_0 \leq (\ell+1)(2\ell+1)^{2w}.$$

Since $x_q x_{q+1}$ is an edge, vertex x_{q+1} has to appear in a bag before index i_q. By using (7) successively, we obtain

$$i_q \leq (q+1)(\ell+1)(2\ell+1)^{2w}.$$

Substituting q with λ in the previous equation and noting that $i_\lambda = L$, we obtain that

$$L \leq (\lambda+1)(\ell+1)(2\ell+1)^{2w}.$$

Since $\lambda = \mathrm{dist}_G(X_0, X_L)$, this concludes the proof of Claim 4.C. ◻

Let us now focus on T_S. Recall that its leaves are a subset of the ancestors of vertices of S. Let A be the set of ancestors and I be the set of inner vertices of degree at least 3 in T_S (note that I has cardinality at most $k-1$ since T_S has at most k leaves). We decompose T_S into (not necessarily disjoint) threads as follows. From any vertex X in $A \cup I$, consider the thread to the closest vertex that is either in I or in A on the unique path from X to the root X_r. Each of these threads satisfies the conditions of Claim 4.C and thus has size bounded above by $(d+1)(\ell+1)(2\ell+1)^{2w}$. Moreover, we have at most $|A|+|I|$ such threads, and $|A|+|I|$ is at most $2k-1$. We can then conclude that

$$|V(T_S)| \leq (2k-1)(d+1)(\ell+1)(2\ell+1)^{2w}.$$

For each bag X in T_S, we may have removed from T a part of the subtree $T(X)$ verifying the hypothesis of Claim 4.B. In the end, the union of all the bags cannot be too large. We then obtain the following upper bound on the order of G:

$$\begin{aligned}|V(G)| &\leq |V(T_S)|\,(d+1)(2\ell+1)^w \\ &\leq (2k-1)(d+1)(\ell+1)(2\ell+1)^{2w}(d+1)(2\ell+1)^w \\ &= \mathcal{O}(kd^2(2\ell+1)^{3w+1}).\end{aligned}$$

This concludes the proof of Theorem 4. ∎

We immediately obtain some corollaries of Theorem 4. The first one is due to the fact that the treelength is trivially upper-bounded by the diameter; for graphs with constant treewidth, it implies the upper bound $n = \mathcal{O}(kd^{\mathcal{O}(1)})$.

COROLLARY 5. *Let G be a graph of treewidth at most w, diameter d, and a resolving set of size k. Then*

$$n = \mathcal{O}(k2^{3w}d^{3w+3}).$$

In particular, if G is K_4-minor-free, then

$$n = \mathcal{O}(kd^9).$$

We have another corollary for chordal graphs, based on the following observation and on the fact that chordal graphs have treelength 1 [15].

Observation 6. If G is a chordal graph of treewidth w and with a resolving set of size k, then $w \leq 3^k$.

Proof. Let v be a vertex of G, and let $x \in N[v]$. For any vertex s in a resolving set of size k of G, there are at most three possible distance values for the distance $d(x,s)$ since any two vertices in $N[v]$ are at distance at most 2. Thus, there can be at most 3^k vertices in $N[v]$, which proves that $\Delta(G) \leq 3^k$. Now, a chordal graph of treewidth w must have a clique of size $w+1$ (indeed, it is well known that in any optimal tree decomposition of a chordal graph, each bag forms a clique); thus, we have $w \leq \Delta(G)$. ∎

COROLLARY 7. *If G is a chordal graph with diameter d, treewidth w, and a resolving set of size k, then*

$$n = \mathcal{O}(kd^2 3^{3w+1}).$$

Moreover,

$$n = \mathcal{O}(d^2 2^{2^{\mathcal{O}(k)}}).$$

We do not know whether the bounds presented in this section are tight. We note that for interval graphs, which are chordal, it is known that a bound of the form $n = \mathcal{O}(dk^2)$ holds, and there are interval graphs for which $n = \Theta(dk^2)$ [17]. By Theorem 2, there are trees that satisfy $n = \Theta(d^2k)$.

3. Graphs of bounded distance-VC dimension. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A *test cover* of $\mathcal{H}$ is a set of edges $\mathcal{C}$ such that each vertex is covered by some edge of $\mathcal{C}$ and for any pair x,y of vertices there is an edge of $\mathcal{C}$ containing exactly one vertex among $\{x, y\}$. We denote by $TC(\mathcal{H})$ the minimum size of a test cover of $\mathcal{H}$. A hypergraph is *twin-free* if for any two distinct vertices, there is at least one hyperedge containing exactly one of them. One can easily check that a hypergraph admits a test cover if and only if it is twin-free. The *projection* of $\mathcal{H}$ on a set X of vertices is defined as $\mathcal{H}_{|X} := \{e \cap X : e \in \mathcal{E}\}$. A set of vertices X is *shattered* in $\mathcal{H}$ if $|\mathcal{H}_{|X}| = 2^{|X|}$. The maximum size of a shattered set in $\mathcal{H}$ is the *VC dimension* of $\mathcal{H}$, denoted by $vc(\mathcal{H})$.

A 2-*shattered set* in a hypergraph $\mathcal{H}$ is a set X such that for all $X' \subset X$ of size 2, there is a hyperedge e such that $e \cap X = X'$. The 2-*VC dimension* of $\mathcal{H}$ is the maximum size of a 2-shattered set in $\mathcal{H}$. Clealry, the 2-VC dimension of $\mathcal{H}$ is at least as large as its VC dimension.

The *dual hypergraph* of a hypergraph $\mathcal{H}$ is denoted $\mathcal{H}^*$: It is the hypergrah whose vertices are the hyperedges of $\mathcal{H}$, and vice versa, and where the incidence relation is the same as in $\mathcal{H}$. The *dual VC dimension* of $\mathcal{H}$ is the VC dimension of the dual and is denoted by $vc^*(\mathcal{H})$. We always have the following inequalities [1]:

$$\log(vc^*(\mathcal{H})) \leq vc(\mathcal{H}) \leq 2^{vc^*(\mathcal{H})}.$$

The following standard lemma is crucial in the study of the VC dimension.

LEMMA 8 (Sauer–Shelah lemma [25, 27]). *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph and X is a subset of vertices, then $|\mathcal{H}_{|X}| \leq |X|^{vc(\mathcal{H})} + 1$.*

3.1. A dichotomy theorem for test covers and VC dimension. If G is a graph, one can define the closed neighborhood hypergraph $\mathcal{H}_1(G)$ of G that has vertex set $V(G)$ and edge set the set of closed neighborhoods of vertices of G. An *identifying code* of G is a test cover of $\mathcal{H}_1(G)$, and the VC dimension of G is often defined as the VC dimension of $\mathcal{H}_1(G)$. A graph G is *twin-free* if $\mathcal{H}_1(G)$ is twin-free. In [9], the VC dimension and identifying codes are related by the following dichotomy result.

THEOREM 9 [9]. *For every hereditary class of graphs $\mathcal{C}$, either*

1. *for every $k \in \mathbb{N}$, there exists a graph $G_k \in \mathcal{C}$ with more than $2^k - 1$ vertices and an identifying code of size $2k$, or*
2. *there exists $\varepsilon > 0$ such that no twin-free graph $G \in \mathcal{C}$ with n vertices has an identifying code of size smaller than n^{ε}.*

We show next that Theorem 9 can be extended to test covers.

PROPOSITION 10. *If $\mathcal{H}$ is a twin-free hypergraph, then*

$$|V| \leq (TC(\mathcal{H}))^{vc^*(\mathcal{H})} + 1.$$

Proof. Let $\mathcal{H}^*$ be the dual hypergraph of $\mathcal{H}$. Let $\mathcal{C}$ be a test cover of $\mathcal{H}$ of size $TC(\mathcal{H})$. We have $|\mathcal{H}^*_{|\mathcal{C}}| = |V|$ since otherwise two vertices of V would belong to the same set of edges of $\mathcal{C}$. Then by Lemma 8, we have $|\mathcal{H}^*_{|\mathcal{C}}| \leq |\mathcal{C}|^{vc(\mathcal{H}^*)} = TC(\mathcal{H})^{vc^*(\mathcal{H})} + 1$. Therefore, $|V| \leq TC(\mathcal{H})^{vc^*(\mathcal{H})} + 1$. ∎

We can also prove the converse.

PROPOSITION 11. *Let $\mathcal{C}$ be a class of hypergraphs that is stable by taking projections. If $\mathcal{C}$ has unbounded dual VC dimension, then for any integer k, there exists a hypergraph $\mathcal{H}$ in $\mathcal{C}$ with $2^k - 1$ vertices and a test cover of size k.*

Proof. Notice first that for any k, $\mathcal{C}$ contains a hypergraph with dual VC dimension exactly k. Indeed, assume that $\mathcal{H}$ is the hypergraph of $\mathcal{C}$ with the smallest dual VC dimension k' larger or equal to k. Then let $\mathcal{A}$ be a shattered set of hyperedges of size k'. Let X be a set of vertices that shatters $\mathcal{A}$ (that is, for each subset of hyperedges of $\mathcal{A}$, there is a unique vertex of X belonging to this subset). Remove one vertex x of X, and let $\mathcal{H}' = \mathcal{H}_{|X-x}$. Then $\mathcal{H}'$ belongs to $\mathcal{C}$ and $vc^*(\mathcal{H}') = k' - 1$. Thus, by our assumptions, $k' = k$.

Now consider a hypergraph of $\mathcal{C}$ with dual VC dimension k, and, as before, let $\mathcal{A}$ be a shattered set of size k and X be a set of 2^k vertices such that for each subset of hyperedges of $\mathcal{A}$, there is exactly one vertex of X that is contained to exactly this subset of hyperedges. Let x_0 be the vertex of X that is contained in no hyperedges, and consider the hypergraph $\mathcal{H}$ induced by $X \setminus \{x_0\}$. Notice first that $\mathcal{H}$ belongs to $\mathcal{C}$. By construction, $\mathcal{H}$ has $2^k - 1$ vertices, and the set of hyperedges of $\mathcal{A}$ forms a test cover. Furthermore, any proper subset of hyperedges is not a test cover since the minimum size of a test cover among $2^k - 1$ vertices is k. ∎

3.2. Metric dimension, VC dimension, and diameter. In contrast to test covers, there is no direct relation between the VC dimension of G and its metric dimension. Indeed, consider the family of line graphs. Any line graph has VC dimension at most 4. Nevertheless, there is a line graph with more than 2^k vertices, diameter 4, and metric dimension at most k. Indeed, consider the following graph. Take k disjoint edges $\{e_1, \ldots, e_k\}$ and $2^k - 1$ disjoint edges $\{e'_I, I \subseteq \{1, \ldots, k\}, I \neq \emptyset\}$ corresponding to the nonempty subsets of $\{e_1, \ldots, e_k\}$. For each edge e'_I, add $|I|$ edges between one endpoint of e'_I (always the same one) and all the endpoints of e_i for $i \in I$ (again, choose always the same endpoint for e_i). Let G be the line graph of this graph. The graph G has $k + 2^k - 1 + \sum_{i=1}^{k} i\binom{k}{i}$ vertices and diameter 4. Moreover, the set S of vertices corresponding to the edges $\{e_1, \ldots, e_k\}$ forms a resolving set. Indeed, a vertex corresponding to an edge e'_I has distance 2 to e_i if $i \in I$ and 4 otherwise. A vertex corresponding to an edge between e'_I and e_i (with $i \in I$) has distance 1 to e_i, 2 to e_j when $j \in I$, and 4 otherwise. Therefore, all the edges have unique distance vector to S.

However, there is such a relation when we consider the *distance-VC dimension*, introduced by Bousquet and Thomassé [10]. The *distance hypergraph* of G is the hypergraph $\mathcal{H}(G)$ with vertex set V and, for all ℓ, all the balls of radius ℓ. The *distance-VC dimension* of G, $dvc(G)$, is the VC dimension of $\mathcal{H}(G)$. The *dual distance-VC dimension* of G, denoted $dvc^*(G)$, is the VC dimension of $\mathcal{H}(G)^*$. Similarly, the *(dual)* 2*-distance VC dimension* of G is the 2-VC dimension of $\mathcal{H}(G)$ ($\mathcal{H}(G)^*$, respectively.

We first give a relation between test covers in $\mathcal{H}(G)$ and the metric dimension of G.

PROPOSITION 12. *If G is a graph of diameter d and metric dimension k, then we have the following:*

$$\frac{TC(\mathcal{H}(G)) - 1}{d} \leq k \leq TC(\mathcal{H}(G)).$$

Proof. Let T be a test cover of $\mathcal{H}(G)$. Then the set of centers of the balls corresponding to the hyperedges of T form a resolving set. Indeed, let $x, y \in V$, and assume without loss of generality that there exists $B \in T$ such that $x \in B$ and $y \notin B$. Let v be the center of B, and let r be its radius. Then v resolves $\{x, y\}$ since $d(v, x) \leq r < d(v, y)$. This shows that $k \leq TC(\mathcal{H}(G))$.

Now let R be a resolving set, and let T be the set of balls centered in vertices of R for all radius from 0 to $d - 1$ plus any ball with radius d. Then T is a test cover of $\mathcal{H}(G))$. Indeed, let $x, y \in V$, and let $z \in R$ such that $d(z, x) \neq d(z, y)$. Assume without loss of generality that $d(z, x) < d(z, y)$. Then $d(z, x) < d$ and the ball centered in z with radius $d(z, x)$ distinguishes x and y. Thus, since any vertex is covered by the ball of radius d, the test cover T has size $d|R| + 1$. ∎

We deduce the following.

PROPOSITION 13. *If G is a graph of order n with diameter d and a resolving set of size k, then*

$$n \leq (dk + 1)^{dvc^*(G)} + 1.$$

Proof. By Proposition 10, $n \leq (TC(\mathcal{H}(G)))^{dvc^*(G)} + 1$. Then, by Proposition 12, we have $TC(\mathcal{H}(G)) \leq kd + 1$. ◻

Proposition 13 is useful when one can bound the dual distance-VC dimension of a graph. The next proposition gives a relation between dvc and dvc^*.

PROPOSITION 14. *If G is a graph of diameter d, then*

$$\frac{1}{\log dvc(G)}(dvc(G) - \log d) \leq dvc^*(G) \leq d \cdot dvc(G).$$

Proof. For the first inequality, let k denote $dvc(G)$. Let S be a shattered set of $\mathcal{H}(G)$ of size k. For each subset X of S, there exists a ball B such that $B \cap S = X$. Let $\mathcal{B}$ be the set of those balls. Among all the radii used in $\mathcal{B}$, let us consider the most used ℓ, and let $\mathcal{B}_\ell$ be the set of balls of $\mathcal{B}$ of radius ℓ. We have $|\mathcal{B}_\ell| \geq \frac{|\mathcal{B}|}{d} \geq \frac{2^k}{d}$. Considering the hypergraph $\mathcal{H}_\ell(G)$ formed by all balls of G of radius ℓ, we have $|\mathcal{H}_\ell(G)_{|S}| \geq |\mathcal{B}_\ell| \geq \frac{2^k}{d}$, and then, by Lemma 8, $\frac{2^k}{d} \leq k^{vc(\mathcal{H}_\ell(G))}$, which implies that $\frac{k-log(d)}{log(k)} \leq vc(\mathcal{H}_\ell(G))$. Now since $\mathcal{H}_\ell$ is isomorphic to its dual, we have $\frac{k-log(d)}{log(k)} \leq vc^*(\mathcal{H}_\ell) \leq vc^*(\mathcal{H}(G)) = dvc^*(G)$.

For the second inequality, let S be a shattered set of $\mathcal{H}(G)^*$ of size $dvc^*(G)$. Let ℓ be the most used radius in S, and let S_ℓ be the set of balls of S of radius ℓ. Let $\mathcal{H}_\ell(G)$ be the the hypergraph formed by all balls of G of radius ℓ. Notice that $\mathcal{H}_\ell(G)$ is isomorphic to its dual $\mathcal{H}_\ell(G)^*$, and then $vc(H_\ell(G)) = vc(H_\ell(G)^*)$. Observe now that S_ℓ is a shattered set of $\mathcal{H}_\ell(G)^*$, and since $|S_\ell| \geq \frac{dvc^*(G)}{d}$, we have $dvc(G) \geq vc(H_\ell(G)) \geq \frac{dvc^*(G)}{d}$. ◻

Bousquet and Thomass proved that graphs of bounded rankwidth[1] and K_t-minor-free graphs have bounded distance 2-VC dimension (and thus bounded distance-VC dimension).

THEOREM 15 [10]. *A K_t-minor-free graph has distance 2-VC dimension at most $t-1$. The distance 2-VC dimension of a graph with rankwidth r is at most $3 \cdot 2^r + 2$.*

Since the distance 2-VC dimension is always larger than the distance-VC dimension and using Proposition 14, we have the following corollaries of Proposition 13.

COROLLARY 16. *Let G be a graph of order n and diameter d and with a resolving set of size k. If K_t is not a minor of G, then*

$$n \leq (dk+1)^{d(t-1)} + 1.$$

If G has rankwidth at most r, then

$$n \leq (dk+1)^{d(3 \cdot 2^r + 2)} + 1.$$

3.3. The dual 2-distance VC dimension and K_t-minor-free graphs. In this section, we improve the bound of Corollary 16 for K_t-minor-free graphs.

THEOREM 17. *If G is a K_t-minor-free graph of diameter d and order n with a resolving set of size k, then $n \leq (dk+1)^{t-1} + 1$.*

To prove Theorem 17, we combine Proposition 13 with the following theorem, which is a "dual" version of Theorem 15. We denote the length of a path P by $\ell(P)$.

[1]We do not define this concept here since we barely use it and refer the reader to [10] instead. Note that any graph of bounded treewidth or cliquewidth also has bounded rankwidth.

THEOREM 18. *If the dual distance 2-VC dimension of a graph G is at least t, then K_t is a minor of G.*

Proof. To prove Theorem 18, we adapt the proof of [10] for distance 2-VC dimension to the dual distance 2-VC dimension and prove the following.

Let $\{(v_1, r_1), \ldots, (v_t, r_t)\}$ be a 2-shattered set in the dual of $\mathcal{H}(G)$. Then, for all i, j, there exists x_{ij} such that

- $d(x_{ij}, v_i) \le r_i$;
- $d(x_{ij}, v_j) \le r_j$;
- $d(x_{ij}, v_k) > r_k$ if $k \notin \{i, j\}$.

For any such x_{ij}, a path formed by a path P between v_i and x_{ij} and a path P' between x_{ij} and v_j such that $\ell(P) \le r_i$ and $\ell(P') \le r_j$ is called a *good ij-path.* For a path P and two vertices x, y in P, we denote by $P[x, y]$ the subpath of P between x and y.

CLAIM 18.A. *If i, j, k, l are distinct and P_{ij}, P_{kl} are two good paths, then $P_{ij} \cap P_{kl} = \emptyset$.*

Proof of claim. Let $P_i := P_{ij}[v_i, x_{ij}]$ be the path from v_i to x_{ij}, and let $P_k := P_{kl}[v_k, x_{kl}]$ be the path from v_k to x_{kl}. Suppose for contradiction that there exists $u \in P_{ij} \cap P_{kl}$. Assume without loss of generality that $u \in P_i \cap P_k$ and that $\ell(P_k[u, x_{kl}]) \le \ell(P_i[u, x_{ij}])$. Then, since $\ell(P_i) = \ell(P_i[v_i, u]) + \ell(P_i[u, x_{ij}]) \le r_i$, we have $d(v_i, x_{kl}) \le \ell(P_i[v_i, u]) + \ell(P_i[u, x_{kl}]) \le r_i$, which is a contradiction. ∎

CLAIM 18.B. *If i, j, k are distinct and P_{ij}, P_{ik} are two good paths that intersect in z, then x_{ij} and x_{ik} cannot both be in the part of P_{ij} (resp., P_{ik}) that is between v_i and z. Hence, at least one of $x_{ij} \in P_{ij}[z, v_j]$ or $x_{ik} \in P_{ik}[z, v_k]$ is true.*

Proof of claim. Suppose, to the contrary, that both x_{ij} and x_{ik} are between z and v_i, and assume without loss of generality that $\ell(P_{ij}[z, x_{ij}]) \le \ell(P_{ik}[z, x_{ik}])$. Then we have

$$\begin{aligned} d(v_k, x_{ij}) &\le \ell(P_{ik}[v_k, z]) + \ell(P_{ij}[z, x_{ij}]) \\ &\le \ell(P_{ik}[v_k, z]) + \ell(P_{ik}[z, x_{ik}]) = \ell(P_{ik}[v_k, x_{ik}]) \le r_k, \end{aligned}$$

contradicting the fact that $d(v_k, x_{ij}) > r_k$. ∎

CLAIM 18.C. *If i, j, k are distinct and P_{ij}, P_{ik} and P_{jk} are three good paths, then $P_{ij} \cap P_{ik} \cap P_{jk} = \emptyset$.*

Proof of claim. Let $z \in P_{ij} \cap P_{ik} \cap P_{jk}$. Assume without loss of generality that

$$\ell(P_{ij}[z, x_{ij}]) = min(\ell(P_{ij}[z, x_{ij}]), \ell(P_{ik}[z, x_{ik}]), \ell(P_{jk}[z, x_{jk}])).$$

Assume furthermore that $x_{ij} \in P_{ij}[v_i, z]$. By Claim 18.B, $x_{ik} \in P_{ik}[z, v_k]$ and $x_{jk} \in P_{jk}[z, v_j]$. Now we have

$$\begin{aligned} d(v_k, x_{ij}) &\le \ell(P_{jk}[v_k, z]) + \ell(P_{ij}[z, x_{ij}]) \\ &\le \ell(P_{jk}[v_k, z]) + \ell(P_{jk}[z, x_{jk}]) = \ell(P_{jk}[v_k, x_{jk}]) \le r_k, \end{aligned}$$

contradicting the fact that $d(v_k, x_{ij}) > r_k$. ∎

For all $x \in V$, we give label i to x if there exists two good paths P_{ij} and P_{ik} that intersect in x. Note that v_i has label i.

Claim 18.D. *For all $x \in V$, x has at most one label.*

Proof of claim. Let P_{ij} and P_{kl} be two good paths containing x. By Claim 18.A, we have $\{i,j\} \cap \{l,k\} \neq \emptyset$. Assume without loss of generality that $i = k$. Assume now that there exists a third good path P_{mn} containing x. We show that $i \in \{m,n\}$. Suppose, to the contrary, that $i \neq m$ and $i \neq n$. Since $x \in P_{ij} \cap P_{mn}$, by Claim 18.A, either $m = j$ or $n = j$ (say $m = j$). Now since $x \in P_{il} \cap P_{mn}$, we have that $n = l$. But then $x \in P_{ij} \cap P_{il} \cap P_{jl}$, which is in contradiction with Claim 18.C. So every good path containing x is a good path from v_i, and then x has only label i. ∎

Let C_i be the set of vertices that are labeled i. Since v_i has label i, C_i is nonempty.

Claim 18.E. *For all $i \leq d$, C_i induces a connected subgraph.*

Proof of claim. We will prove that for each vertex $u \in C_i$, there exists a path in C_i from u to v_i. Assume that $u \in P_{ij} \cap P_{il}$. By Claim 18.B, either $x_{ij} \in P_{ij}[v_i, u]$ or $x_{il} \in P_{il}[v_i, u]$. Assume without loss of generality that $x_{ij} \in P_{ij}[v_i, u]$. By definition of x_{il}, $r_j < d(v_j, x_{il}) \leq \ell(P_{ij}[u, v_j]) + \ell(P_{il}[u, x_{il}])$, and since $\ell(P_{ij}[u, v_j]) + \ell(P_{ij}[x_{ij}, u]) \leq r_j$, we have $\ell(P_{il}[u, x_{il}]) > \ell(P_{ij}[x_{ij}, u])$. Then, since $\ell(P_{il}[v_i, u]) + \ell(P_{il}[u, x_{il}]) = \ell(P_{il}[v_i, x_{il}]) \leq r_i$, we have $\ell(P_{il}[v_i, u]) + \ell(P_{ij}[u, x_{ij}]) \leq r_i$. Thus, $P_{il}[v_i, u] \cup P_{ij}[u, x_{ij}] \cup P_{ij}[x_{ij}, v_j]$ is a good ij-path. We conclude that all vertices of $P_{il}[v_i, u]$ have label i (that is, $P_{il}[v_i, u] \subseteq C_i$). ∎

Assume now that $\{(v_1, r_1), \ldots, (v_t, r_t)\}$ is a 2-shattered set in the dual of $\mathcal{H}(G)$. Then the sets C_i form nonempty connected disjoint sets of vertices, and there are disjoint paths between any pair of such sets. Thus, there is a minor K_t, completing the proof of Theorem 18. ∎

3.4. Outerplanar graphs. Outerplanar graphs are K_4-minor-free and have treewidth at most 2. Hence, by Theorem 17, $n = \mathcal{O}(d^3k^3)$, and by Corollary 5, $n = \mathcal{O}(d^9k)$. We will improve these bounds using a different method.

Theorem 19. *If G is an outerplanar graph with diameter d and a resolving set of size k, then G has order at most $2kd^2 - 2d^2 + d + 1 = \mathcal{O}(kd^2)$.*

Proof. Let S be a resolving set of G of size k, and let $s_1 \in S$. We consider a circular layout of G, that is, a planar representation of G with all the vertices lying on the boundary of a circle $\mathcal{C}$ (it is not difficult to see that such a layout exists; see [28]). The vertices of G can be naturally ordered following $\mathcal{C}$ and starting by s_1. We denote this order by $<$.

Claim 19.A. *Let $x < y < z < t$ be four vertices of G. Let P_1 be a path from y to t and P_2 be a path from x to z. Then P_1 and P_2 must intersect.*

Proof of claim. Indeed, the drawing of the path P_2 cuts the disk formed by $\mathcal{C}$ into two disjoint components, and the vertices y and t are not in the same component. Therefore, the drawing of the path P_1 must intersect P_2, and since the representation is planar, it must be on a vertex. ∎

For each $1 \leq i \leq d$, we define L_i to be the set of vertices at distance exactly i of s_1. The following claim is key to our proof.

Claim 19.B. *Let $i \in \{1, \ldots, d\}$. Let $s \in S$ and y a vertex of L_i that minimizes the distance between s and vertices of L_i. Let u and v be two vertices of L_i. If $y < u < v$ or $v < u < y$, then $d(s, u) \leq d(s, v)$.*

Proof of claim. We assume that $s_1 < y < u < v$ (the other case is symmetric).

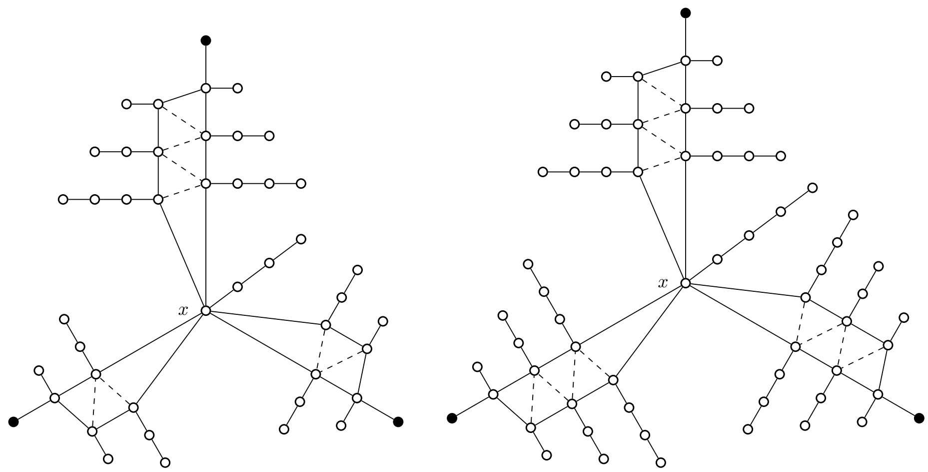
We first prove that $d(y, u) \leq d(y, v)$. Let P_1 be a shortest path from y to v and P_2 be a shortest path from u to s_1. By Claim 19.A, P_1 and P_2 must intersect in some vertex z. Since P_2 is a shortest path from u to s_1, it has length i and $d(z, s_1) \leq i$, and so $z \in L_j$ with $j \leq i$. Furthermore, we have $d(z, v) \geq i - j = d(z, u)$. Let P be the path from y to u that consists of the subpath of P_1 from y to z, followed by the subpath of P_2 from z to u. Since $d(z, u) \leq d(z, v)$, the path P is not longer than P_1, and thus $d(y, u) \leq d(y, v)$.

This proves the claim when $s \in L_i$. Assume now that $s \in L_j$ and that $j < i$. Let P_1 be a path formed by the union of a shortest path $P_{1,1}$ from y to s and a shortest path $P_{1,2}$ from s to v. Let P_2 be a shortest path from u to s_1. By Claim 19.A, P_1 and P_2 must intersect in z, and, as before, $z \in L_k$ with $k \leq i$. If z belongs to $P_{1,1}$, that is, to a shortest path between y and s, then the path from s to u following $P_{1,1}$ until z and then P_2 until u is shorter than $P_{1,1}$ (since $d(z, u) \leq d(z, y)$). Hence, $d(s, u) \leq d(s, y)$, but y is minimizing the distance between s and a vertex of L_i. Therefore, $d(s, y) = d(s, u) \leq d(s, v)$. Otherwise, z must belong to $P_{1,2}$, a shortest path between s and v. Then the path P from s to u that follows $P_{1,2}$ until z and then P_2 until u is shorter than $P_{1,2}$. Indeed, since $z \in L_k$, we have $d(z, u) = i - k$ and $d(z, v) \geq i - k$. Thus, $d(s, u) \leq d(s, v)$.

Assume finally that $s \in L_j$ with $j > i$. We have $d(s, y) = j - i$ (indeed, a shortest path from s to s_1 must pass by a vertex $y' \in L_i$ and then $d(s, y') = j - i$). Let P_1 be a path formed by the union between a shortest path $P_{1,1}$ from y to s and a shortest path $P_{1,2}$ from s to v. Let P_2 a shortest path from u to s_1. Again, P_1 and P_2 must intersect in $z \in L_k$ with $k \leq i$. Since $P_{1,1}$ is a path of length $j - i$ between L_i and L_j, all the vertices of $P_{1,1}$ are in a layer $L_{j'}$ with $i \leq j' \leq j$. It is not possible to have $z = y$ since in P_2 there is exactly one vertex by $L_{j'}$ for $j' \leq i$, and $u \neq y$ is this vertex for $j' = i$. Hence, z is in $P_{2,2}$. It means that there is a vertex $z' \in L_i$ on the path from s to z: Indeed, when going from L_j to L_k, a path must intersect all the layers between L_k and L_j. We choose for z' the first vertex of L_i we meet on $P_{1,2}$ going from s to v. If $z' < u < v$, then as in the first case of the proof, $d(z', u) \leq d(z', v)$ and so $d(s, u) \leq d(s, v)$ and we are done. Otherwise, we have $s_1 < y < u < z'$, and there is a path from y to z' (the path P_1 stopped in z') that is not intersecting a path from s_1 to u, which contradicts Claim 19.A. This completes the proof of Claim 19.B. $\square$

We can now finish the proof of Theorem 19. By Claim 19.B, each vertex $s \neq s_1$ of S partitions the vertices of L_i with respect to the order $<$ into at most $2d+1$ parts such that two vertices belonging to the same part have the same distance to s. Hence, together, the vertices of $S \setminus \{s_1\}$ partition L_i into at most $2d(k-1)+1$ parts, and the distance to S of each vertex of L_i is determined by its position in the partition. Hence, there is at most one vertex in each part, and thus $|L_i| \leq 2d(k-1)+1$. Finally, the total number of vertices of G is at most $1 + \sum_{i=1}^{d} |L_i| \leq 2d^2(k-1) + d + 1$. This completes the proof of Theorem 19. $\square$

We now show that Theorem 19 is tight, up to a constant factor. For two integers $d, k \geq 2$, let $O_{d,k}$ be the outerplanar graph constructed as follows. First, for some integer i, we define a graph H_i as follows. Consider a cycle C of length $2i + 1$, where x is a distinguished vertex of C. To any vertex v of C at distance $j \geq 1$ of x in C, we attach a path of length $i - j + 1$ to v, and to one of the two vertices at distance i of x in C, we attach a second leaf. Now, $O_{d,k}$ is built from $k - 1$ copies of $H_{\lfloor d/2 \rfloor - 1}$ and one copy of $H_{\lceil d/2 \rceil - 1}$ identified at x, with an additional path of length $\lfloor d/2 \rfloor$ attached to x. (Note that we may optionally add chords to the cycles in $O_{d,k}$, as long as the outerplanarity and the distances from each vertex having two leaves attached are preserved.) The order of $O_{d,k}$ is $\frac{d+2}{2} + k(2\sum_{i=1}^{d/2} i - 1)$ when d is even and $\frac{3d+3}{2} + k(2\sum_{i=1}^{\lfloor d/2 \rfloor} i - 1)$ when d is odd; this is $(\frac{1}{4} + o(1))kd^2$. See the graph of Figure 2 for an illustration of $O_{7,3}$ and $O_{8,3}$.



(a) The graph $O_{7,3}$.

(b) The graph $O_{8,3}$.

FIG. 2. *The graphs $O_{7,3}$ and $O_{8,3}$. Dashed edges are optional. Black vertices form an optimal resolving set.*

PROPOSITION 20. *Let $d, k \geq 2$ be two integers. The outerplanar graph $O_{d,k}$ has diameter d, metric dimension k, and order $(\frac{1}{4} + o(1))kd^2$.*

Proof. The values of the diameter and the order follow from the definition. To see that the metric dimension is k, consider the k vertices that have two neighbors of degree 1. In order for these two neighbors to be distinguished, one of them needs to be in any resolving set. Now, we pick exactly one of them and repeat this for every such pair; we obtain a set S of k vertices. We claim that S is a resolving set. Let $S = \{s_1, \ldots, s_k\}$, and let us call C_i the component of $O_{d,k} - x$ containing s_i. (There are $k + 1$ components in $O_{d,k} - x$, with one of them isomorphic to a path and not containing any vertex of S.) Any two vertices u and v from different components of $O_{d,k} - x$ are distinguished since at least one of them (say u) has a vertex s_i of S in its component C_i and $d(u, s_i) < d(v, s_i)$. Within a component C of $O_{d,k} - x$, vertices with distinct distances to x are distinguished by the vertices of S not in C. Finally, vertices at the same distance of x in a component C_i are distinguished by s_i. ∎

4. Conclusion.

For trees and outerplanar graphs, we know that $n = \mathcal{O}(kd^2)$, and this is tight. We do not know whether our other bounds are tight. It would be interesting to further study the classes of graphs of fixed treewidth w (or the more restricted case of w-trees) and the class of chordal graphs. We have proved that $n = \mathcal{O}(kd^{3w+3})$ for constant w (Corollary 5) and $n = \mathcal{O}(f(k)d^2)$ for chordal graphs, where f is doubly exponential (Corollary 7). Can these bounds be improved? Moreover, Corollary 16 gives a bound in terms of rankwidth. Trying to get a similar result in terms of cliquewidth seems to be a natural follow-up.

Another interesting problem is to determine the best possible bound for planar graphs, that is, whether our $n = \mathcal{O}(d^4k^4)$ bound that follows from Theorem 17 can be improved. Note that $n = \mathcal{O}(d^2)$ holds when the metric dimension is 2; indeed, in this case, we have $n \leq d^2 + 2$ for any graph [12, 21]. This quadratic bound is matched by any square grid, which has metric dimension 2 and $n = d^2$. Nevertheless, there are planar graphs with metric dimension 3 and order $\Theta(d^3)$. Such a family of graphs can be described as follows. Pick any integer t and consider t disjoint copies $G_1, G_2, \ldots G_t$ of a $t \times t$ grid. For i between 1 and $t - 1$, add an edge between the top left corners of G_i and G_{i+1} and another edge between the top right corners of G_i and G_{i+1}. The diameter of this graph is $4t$, and its order is t^3. Moreover, the top corners of G_1 together with the bottom left corner of G_t form a resolving set of size 3. We do not know whether there are planar graphs with small metric dimension and order $\Theta(d^4)$.

For the smaller class of treewidth 2 graphs (that is, K_4-minor-free graphs), we know that $n = \mathcal{O}(d^3k^3)$ (Theorem 17) and $n = \mathcal{O}(d^9k)$ (Corollary 5), but we doubt that these bounds are optimal. We remark that our proof method for outerplanar graphs does not seem to be easily generalizable to this class.

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