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Identification, location–domination and metric dimension on interval and permutation graphs. I. Bounds

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ABSTRACT

We consider the problems of finding optimal identifying codes, (open) locating–dominating sets and resolving sets of an interval or a permutation graph. In these problems, one asks to find a subset of vertices, normally called a *solution* set, using which all vertices of the graph are distinguished. The identification can be done by considering the neighborhood within the solution set, or by employing the distances to the solution vertices. Normally the goal is to minimize the size of the solution set then. Here we study the case of interval graphs, unit interval graphs, (bipartite) permutation graphs and cographs. For these classes of graphs we give tight lower bounds for the size of such solution sets depending on the order of the input graph. While such lower bounds for the general class of graphs are in logarithmic order, the improved bounds in these special classes are of the order of either quadratic root or linear in terms of number of vertices. Moreover, the results for cographs lead to linear-time algorithms to solve the considered problems on inputs that are cographs.

1. Introduction

Identification problems in discrete structures are a well-studied topic. In these problems, we are given a graph or a hypergraph, and we wish to distinguish (i.e. uniquely identify) its vertices using (small) set of selected elements from the (hyper)graph. For the *metric dimension*, one seeks a set S of vertices of a graph G where every vertex of G is uniquely identified by its distances to the vertices of S. The notions of *identifying codes* and *(open) locating–dominating sets* are similar; instead of the distances to S, we ask for the vertices to be distinguished by their neighborhood within S. These concepts are studied by various authors since the 1970s and 1980s, and have been applied to various areas such as network verification [3,5], fault-detection in networks [27,37], graph isomorphism testing [2] or the logical definability of graphs [28].

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We note that the related problem of finding a *test cover* of a hypergraph (where hyperedges are selected to distinguish the vertices) has been studied under several names by various authors, see e.g. [8,9,13,21,30].

In this paper, we study identifying codes, (open) locating–dominating sets and the metric dimension of interval graphs, permutation graphs and some of their subclasses. In particular, we study bounds on the order for such graphs with given size of an optimal solution.

Important concepts and definitions. All considered graphs are finite and simple. We will denote by $N[v]$, the *closed neighborhood* of a vertex v, and by $N(v)$ its *open neighborhood* $N[v] \setminus \{v\}$. A vertex is *universal* if it is adjacent to all the vertices of the graph. A set S of vertices of G is a *dominating set* if for every vertex v of G, there is a vertex x in $S \cap N[v]$. It is a *total dominating set* if instead, $x \in S \cap N(v)$. In the context of (total) dominating sets we say that a vertex x *(totally) separates* two distinct vertices u, v if it (totally) dominates exactly one of them. A set S (totally) separates the vertices of a set X if all pairs of X are (totally) separated by a vertex of S. Whenever it is clear from the context, we will only say "separate" and omit the word "totally". We have the three key definitions, that merge the concepts of (total) domination and (total) separation:

Definition 1 *(Slater [33,34]).* A set S of vertices of a graph G is a *locating–dominating set* if it is a dominating set and it separates the vertices of $V(G) \setminus S$. The smallest size of a locating–dominating set of G is the *location–domination number* of G, denoted $\gamma^{\mathrm{LD}}(G)$. Without the domination constraint, this concept has also been used under the name *distinguishing set* in [2] and *sieve* in [28].

Definition 2 *(Karpovsky, Chakrabarty and Levitin [27]).* A set S of vertices of a graph G is an *identifying code* if it is a dominating set and it separates all vertices of $V(G)$. The smallest size of an identifying code of G is the *identifying code number* of G, denoted $\gamma^{\mathrm{ID}}(G)$.

Definition 3 *(Seo and Slater [31]).* A set S of vertices of a graph G is an *open locating–dominating set* if it is a total dominating set and it totally separates all vertices of $V(G)$. The smallest size of an open locating–dominating set of G is the *open location–domination number* of G, denoted $\gamma^{\mathrm{OLD}}(G)$. This concept has also been called *identifying open code* in [25].

Separation could also be done using distances from the members of the solution set. Let $d(x, u)$ denote the distance between two vertices x and u.

Definition 4 *(Harary and Melter [24], Slater [32]).* A set B of vertices of a graph G is a *resolving set* if for each pair u, v of distinct vertices, there is a vertex x of B with $d(x, u) \neq d(x, v)$.[3] The smallest size of a resolving set of G is the *metric dimension* of G, denoted $dim(G)$.

It is easy to check that the inequalities $dim(G) \leq \gamma^{\mathrm{LD}}(G) \leq \gamma^{\mathrm{ID}}(G)$ and $\gamma^{\mathrm{LD}}(G) \leq \gamma^{\mathrm{OLD}}(G)$ hold, indeed every locating–dominating set of G is a resolving set, and every identifying code (or open locating–dominating set) is a locating–dominating set. Moreover it is proved that $\gamma^{\mathrm{ID}}(G) \leq 2\gamma^{\mathrm{LD}}(G)$ [22] (using the same proof idea one would get a similar relation between $\gamma^{\mathrm{LD}}(G)$ and $\gamma^{\mathrm{OLD}}(G)$ and between $\gamma^{\mathrm{ID}}(G)$ and $\gamma^{\mathrm{OLD}}(G)$, perhaps with a different constant factor).

In a graph G of diameter 2, one can easily see that the concepts of resolving set and locating–dominating set are almost the same, as $\gamma^{\mathrm{LD}}(G) \leq dim(G) + 1$. Indeed, let S be a resolving set of G. Then all vertices in $V(G) \setminus S$ have a distinct neighborhood within S. There might be (at most) one vertex that is not dominated by S, in which case adding it to S yields a locating–dominating set.

While a resolving set and a locating–dominating set exist in every graph G (for example the whole vertex set), an identifying code may not exist in G if it contains *twins*, that is, two vertices with the same closed neighborhood. However, if the graph is *twin-free*, then the set $V(G)$ is an identifying code of G. Similarly, a graph admits an open locating–dominating set if and only if it has no *open twins*, vertices sharing the same open neighborhood. We say that such a graph is *open twin-free*.

The focus of this work is to study these concepts and corresponding decision problems for specific subclasses of perfect graphs. Many standard graph classes are perfect, for example bipartite graphs, split graphs, interval graphs. For precise definitions, we refer to the book of Brandstädt, Le and Spinrad [11]. Some of these classes are defined using a geometric intersection model, that is, the vertices are associated to the elements of a set S of (geometric) objects, and two vertices are adjacent if and only if the corresponding elements of S intersect. The graph defined by the intersection model of S is its *intersection graph*. An *interval graph* is the intersection graph of intervals of the real line, and a *unit interval graph* is an interval graph whose intersection model contains only (open) intervals of unit length. Given two parallel lines B and T, a *permutation graph* is the intersection graph of segments of the plane which have an endpoint on B and an endpoint on T. A *cograph* is a graph which can be built from single vertices using the repeated application of two binary graph operations:

[3] Resolving sets are also known under the name of *locating sets* [32]. Optimal resolving sets have sometimes been called *metric bases* in the literature; to avoid an inflation in the terminology we will only use the term *resolving set*.

the disjoint union $G \oplus H$, and the complete join $G \bowtie H$ (another standard characterization of cographs is that they are those graphs that do not contain a 4-vertex-path as an induced subgraph). All cographs are permutation graphs.

Interval graphs and permutation graphs are classic graph classes that have many applications and are widely studied. They can be recognized efficiently, and many combinatorial problems have simple and efficient algorithms for these classes.

Previous work. It is not difficult to observe that a graph G with n vertices and an identifying code or open locating–dominating set S of size k satisfies $n \leq 2^k - 1$ [27,31]. Furthermore it can be observed that this bound is tight. If S is a locating–dominating set, then a tight bound is $n \leq 2^k + k - 1$ [34]. These bounds are tight, even for bipartite graphs or split graphs. They are also tight up to a constant factor for co-bipartite graphs. On the other hand, tight bounds of the form $n = O(k)$ are given for paths and cycles [6,34], trees [7,33] and planar graphs and some of their subclasses [35]. A bound of the form $O(k^2)$ was given for identifying codes in line graphs [19].

The number of vertices of a graph with metric dimension k cannot be bounded by a function of k: for example, an end point of a path (of any length) forms a resolving set. More generally, for every integer k, one can construct arbitrarily large trees with metric dimension k (consider for example a vertex x with $k+1$ arbitrarily long disjoint paths starting from x). However, when the diameter of G is at most D and $dim(G) = k$, we have the (trivial) bound $n \leq D^k + k$ [14], which is not tight but a more precise (and tight) bound is given in [26].

Regarding the algorithmic study of these problems, Identifying Code, Locating–Dominating-Set, Open Locating–Dominating Set and Metric Dimension (the decision problems that ask, given a graph G and an integer k, for the existence of an identifying code, a locating–dominating set, an open locating–dominating set and a resolving set of size at most k in G, respectively) were shown to be NP-complete, even for many restricted graph classes. We refer to e.g. [1,12,16–19,21,29,31] for some results. On the positive side, Identifying Code, Locating–Dominating-Set and Open Locating–Dominating Set are linear-time solvable for graphs of bounded clique-width (using Courcelle's theorem [15]). Furthermore, Slater [33] and Auger [1] gave explicit linear-time algorithms solving Locating–Dominating-Set and Identifying Code, respectively, in trees. Epstein, Levin and Woeginger [17] also gave polynomial-time algorithms for the weighted version of Metric Dimension for paths, cycles, trees, graphs of bounded cyclomatic number, cographs and partial wheels. Diaz, Pottonen, Serna, Jan van Leeuwen [16] gave a polynomial-time algorithm for outerplanar graphs. In a companion paper [20], we prove that all four problems Identifying Code, Locating–Dominating-Set, Open Locating–Dominating Set and Metric Dimension are NP-complete, even for interval graphs and permutation graphs. We also give in [20] an $f(k)poly(n)$-time (i.e. fixed-parameter-tractable) algorithm to check whether an interval graph has metric dimension at most k.

Our results and structure of the paper. In this paper, we give new upper bounds on the maximum order of interval or permutation graphs (and some of their subclasses) having an identifying code, an (open) locating–dominating set or a resolving set of size k. For the three first problems (in which the identification is neighborhood-based), the bounds are $O(k^2)$ for interval graphs and permutation graphs and $O(k)$ for unit interval graphs, bipartite permutation graphs and cographs. We also study the metric dimension of such graphs by giving similar upper bounds in terms of the solution size k and the diameter D. We obtain the bounds $O(Dk^2)$ for interval and permutation graphs, and $O(Dk)$ for unit interval graphs and cographs. We also provide constructions showing that all our bounds are nearly tight. Finally, we give a linear-time algorithm for Identifying Code and Open Locating–Dominating Set in cographs.[4]

Section 2 is devoted to interval graphs, Section 3 to unit interval graphs, Section 4 to permutation graphs, Section 5 to bipartite permutation graphs, and Section 6 to cographs. We conclude the paper in Section 7.

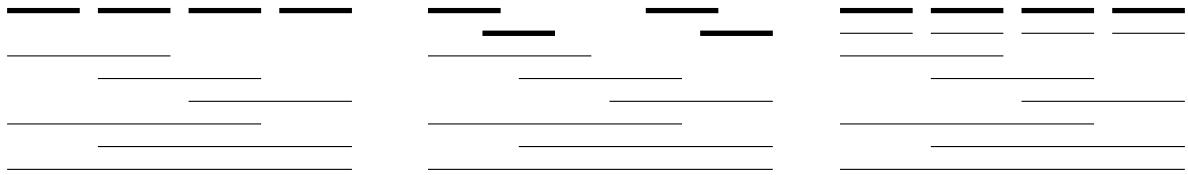
2. Interval graphs

We now give bounds for interval graphs. Recall that in general there are graphs with (open) location–domination or identifying code number k and $\Theta\left(2^k\right)$ vertices [27,34]. This can be improved for interval graphs as follows.

Theorem 5. *Let G be an interval graph on n vertices and let S be a subset of vertices of size k. If S is an open locating–dominating set or an identifying code of G, then $n \leq \frac{k(k+1)}{2}$. If S is a locating–dominating set of G, then $n \leq \frac{k(k+3)}{2}$. Hence, $\gamma^{ID}(G) \geq \sqrt{2n + \frac{1}{4}} - \frac{1}{2}$, $\gamma^{OLD}(G) \geq \sqrt{2n + \frac{1}{4}} - \frac{1}{2}$ and $\gamma^{LD}(G) \geq \sqrt{2n + \frac{9}{4}} - \frac{3}{2}$.*

Proof. Let $S = \{x_1, \ldots, x_k\}$ be an identifying code or open locating–dominating set of G of size k, where the intervals $x_1, \ldots, x_k$ are ordered increasingly by their right endpoint (let us denote by r_i, the right endpoint of interval x_i). Using this order, we define a partition $\mathcal{E}_1, \ldots, \mathcal{E}_k$ of $V(G)$ as follows. Let $\mathcal{E}_1$ be the set of intervals that start strictly before r_1. For any i with $2 \leq i \leq k-1$, let $\mathcal{E}_i$ be the set of intervals whose left endpoint lies within $[r_{i-1}, r_i[$, and let $\mathcal{E}_k$ be the set of intervals whose left endpoint is at least r_{k-1}. Now, let I be an interval of $\mathcal{E}_i$ with $1 \leq i \leq k$. Interval I can only intersect intervals of S in $x_i, \ldots, x_k$.[5] These intervals must be consecutive when considering the order defined by the left endpoints and I must

[4] Remark that the algorithm of Epstein, Levin and Woeginger [17] for Metric Dimension can also be used for Locating–Dominating-Set.
[5] We use a representation with open intervals.



(a) Identifying code (b) Open locating-dominating set (c) Locating-dominating set

Fig. 1. Examples of interval graphs from Proposition 6 reaching the lower bounds of Theorem 5. Solution intervals are in bold.

intersect the first one. There are $k-i+1$ possible intersections and so $\mathcal{E}_i$ contains at most $k-i+1$ intervals. Hence, in total G has at most $\sum_{i=1}^{k}(k-i+1) \leq \frac{k(k+1)}{2}$ vertices.

If S is a locating–dominating set, we reason similarly, but we must take into account the existence of k additional vertices that do not need to be separated (the ones from S).

The bounds on parameters γ^{ID}, γ^{LD} and γ^{OLD} follow directly by using the facts that $k(k+1) = (k+\frac{1}{2})^2 - \frac{1}{4}$ and $k(k+3) = (k+\frac{3}{2})^2 - \frac{9}{4}$. □

Proposition 6. *The bounds of Theorem 5 are tight for every $k \geq 1$.*

Proof. For identifying codes, consider the interval graph formed by the intersection of the following family of intervals: $\mathcal{F} = \{]i, j[\mid 1 \leq i < j \leq k+1, i, j \in \mathbb{N}\}$, where the subfamily $\{]i, i+1[\mid 1 \leq i \leq k\}$ forms an identifying code S of size k. A similar construction can be done for open locating–dominating sets when k is even by replacing the $k/2$ intervals $]2i, 2i+1[$ by intervals $]2i-0.5, 2i+0.5[$. For locating–dominating sets, consider $\mathcal{F}$ with a copy of each interval $]i, i+1[$ of S. Then S is a locating–dominating set. An illustration of these examples for $k = 4$ is given in Fig. 1. □

We now give a bound similar to the one of Theorem 5 for the metric dimension using the diameter and the order of the graph. Recall that in general there are graphs with metric dimension k, diameter D and order $\Theta\left(D^k\right)$ [26].

Theorem 7. *Let G be a connected interval graph on n vertices, of diameter D, and a resolving set of size k. Then $n \leq 2k^2D + 4k^2 + kD + 5k + 1 = \Theta(Dk^2)$.*

Proof. Let S be a resolving set of size k of G and let $s_1, \ldots, s_k$ be the elements of S. For each i in $\{1, \ldots, k\}$, we define an ordered set $L^i = \{x_1^i > x_2^i > \ldots > x_s^i\}$, in the following way. Let x_1^i be the left endpoint of s_i. Assuming x_j^i is defined, let x_{j+1}^i be the smallest among all left endpoints of the intervals of G that end strictly after x_j^i. We stop the process when we have $x_{s+1}^i = x_s^i$, which means that, since G is connected, x_s^i is the smallest left endpoint among all the intervals of G. Note that an interval whose right endpoint lies within $]x_{j+1}^i, x_j^i]$ is at distance exactly $j+1$ of s_i. Furthermore, there is no interval whose right endpoint is smaller than x_s^i.

We similarly define the ordered set $R^i = \{y_1^i < y_2^i < \ldots < y_{s'}^i\}$: y_1^i is the right endpoint of s_i, y_{j+1}^i is the largest right endpoint among all the intervals of G that start strictly before y_j^i, and $y_{s'}^i$ is the largest right endpoint among all the intervals of G. An interval whose left endpoint is within $[y_j^i, y_{j+1}^i[$ is at distance exactly $j+1$ of s_i and no interval has left endpoint larger than $y_{s'}^i$.

Note that intervals at distance 1 of s_i in G are exactly the intervals starting before y_1^i and finishing after x_1^i. More generally, for any interval of G, its distance to s_i is uniquely determined by the position of its right endpoint in the ordered set L^i and the position of its left endpoint in the ordered set R^i. Moreover the interval I_s that defines the point x_s^i of L^i and the interval $I_{s'}$ that defines the point $y_{s'}^i$ of R^i are at distance at least $s+s'-4$ from each other. Indeed, a shortest path from I_s to $I_{s'}$ contains s_i or a neighbor J of s_i. In the best case, J is the interval $]x_2^i, y_2^i[$ and then $d(I_s, I_{s'}) = d(I_s, J) + d(J, I_{s'}) \leq s-2+s'-2$. Therefore, we have $s+s'-4 \leq D$ and $L^i \cup R^i$ contains at most $D+4$ points.

Consider now the union of all the sets $L^i \cup R^i$. Each of these sets has at most $D+4$ points and they all have two common points at the extremities. Thus the union contains at most $k(D+2)+2$ distinct points on the real line and thus defines a natural partition $\mathcal{P}$ of $\mathbb{R}$ into at most $k(D+2)+1$ intervals (we do not count the intervals before and after the extremities since no intervals can end or start there). Any interval of $V(G) \setminus S$ is uniquely determined by the positions of its endpoints in $\mathcal{P}$. Let $I \in V(G) \setminus S$. For a fixed i, by definition of the sets L^i, the interval I cannot contain two points of L^i and similarly, it cannot contain two points of R^i. Thus, I contains at most $2k$ points of the union of all the sets L^i and R^i. Therefore, if P denotes a part of $\mathcal{P}$, there are at most $2k+1$ intervals with left endpoints in P. In total, there are at most $(k(D+2)+1) \cdot (2k+1)$ intervals in $V(G) \setminus S$ and

$$\begin{aligned} |V(G)| &\leq (k(D+2)+1) \cdot (2k+1) + k \\ &= 2k^2D + 4k^2 + kD + 5k + 1. \qquad \square \end{aligned}$$

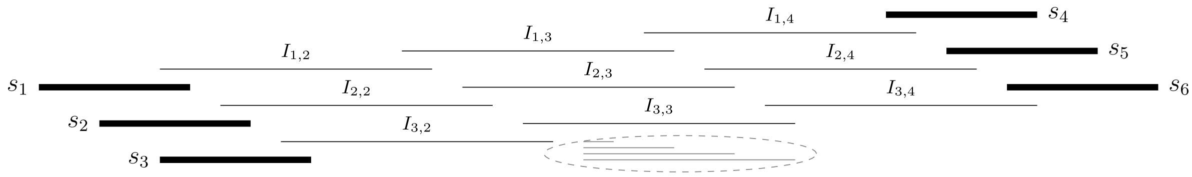


Fig. 2. An interval graph of Proposition 8, reaching the order of the lower bound of Theorem 7. The construction is done for diameter $D = 5$ and resolving set size $k = 6$. The intervals inside the dashed ellipse are the intervals that are added after the end of $I_{3,2}$. Similar intervals are added after the end of each interval $I_{i,j}$ for $2 \le j \le 3$. The intervals $\{s_1, \dots, s_6\}$ (in bold) form a resolving set.

The bound of Theorem 7 is tight up to a constant factor:

Proposition 8. *For every $k \ge 1$ and $D \ge 2$, there exists an interval graph with diameter D, a resolving set of size k, and $\Theta(Dk^2)$ vertices.*

Proof. Assume that k is even (a similar construction can be done if k is odd) and $D \ge 2$. Let $L > k/2$. For $i \in \{1, \dots, k/2\}$ and $j \in \{1, \dots, D\}$, we define the interval $I_{i,j} =](j-1)L + i, jL + 1/2 + i[$. The intervals $I_{i,j}$ for a fixed i induce a path of length $D - 1$. See Fig. 2 for an illustration with $k = 6$ and $D = 5$.

Let $s_i = I_{i,1}$ for $1 \le i \le k/2$ and $s_i = I_{i-k/2,D}$ for $k/2 < i \le k$. Using the notations of the proof of Theorem 7, one can note that, if $1 \le i \le k/2$, then $y^i_j = jL + 1/2 + i$ and if $k/2 < i \le k$, then $x^i_j = (j-1)L + (i - k/2)$.

In particular for $1 \le i \le k/2$ and $1 < j < D$ we have:

$$d(I_{i,j}, s_{i'}) = \begin{cases} j-1 & \text{if } i \le i' \\ j & \text{if } i > i' \end{cases}$$

and, for $k/2 < i \le k$ and $1 < j < D$:

$$d(I_{i,j}, s_{i'}) = \begin{cases} D-j & \text{if } i \ge i' + k/2 \\ D-j+1 & \text{if } i < i' + k/2. \end{cases}$$

Therefore, the set of intervals $\mathcal{S} = \{s_i, 1 \le i \le k\}$ is a resolving set.

We add some intervals that do not influence the shortest paths between the intervals $I_{i,j}$ (in particular, the distances from $I_{i,j}$ to $\mathcal{S}$ do not change). First note that all the intervals $I_{i,j}$ have the same length. Thus there is a natural order on these intervals which is actually defined by $I_{i,j} < I_{i',j'}$ if and only if $j < j'$ or $j = j'$ and $i < i'$. In particular, any set of $k/2$ intervals that are consecutive for this order do not contain two intervals $I_{i,j}$ and $I_{i',j'}$ with $i = i'$.

Consider a particular interval $J = I_{i,j}$ with $2 \le j \le D - 2$. We add $k/2 + 1$ intervals after the end of J in the following way. Consider the set $\{J_0 < J_1 < \dots < J_{k/2}\}$ of the first $k/2 + 1$ intervals starting after the end of J. Note that J_0 and $J_{k/2}$ correspond to a pair of intervals $I_{i,j}$, $I_{i',j'}$ with $i = i'$. For each interval J_s, add an interval starting between the end of J and the beginning of J_0 and finishing before the beginning of J_s and after the beginning of J_{s-1} if $s \ne 0$. See Fig. 2 for an illustration of the intervals added in a particular example ($J = I_{3,2}$). These intervals are all finishing before the end of $J_{k/2}$ and thus are not changing the shortest paths and the values of x^i_j and y^i_j.

All the intervals added this way have distinct distances to set $\mathcal{S}$. Indeed, either they are starting between two different consecutive pairs y^i_j or finishing between different consecutive pairs x^i_j. There are in total $kD + (k/2+1)(D-2)k/2 = \Theta(Dk^2)$ intervals in this graph and its diameter is D. $\square$

3. Unit interval graphs

Using similar ideas as for Theorems 5 and 7, we are able to give improved bounds for unit interval graphs.

Theorem 9. *Let G be a unit interval graph on n vertices and let S be a subset of vertices of size k. If S is an open locating–dominating set or an identifying code of G, then $n \le 2k - 1$. If S is a locating–dominating set of G, then $n \le 3k - 1$. Hence, $\gamma^{ID}(G) \ge \frac{n+1}{2}$, $\gamma^{OLD}(G) \ge \frac{n+1}{2}$, and $\gamma^{LD}(G) \ge \frac{n+1}{3}$.*

Proof. We consider a representation of G with open unit intervals and we denote by ℓ_I and r_I the endpoints of the interval I. Consider an identifying code or open locating–dominating set S of size k. Consider the set of points $T = \{\ell_I - 1, \ell_I + 1, \text{for all } I \in S\}$, and sort T by increasing order: $T = \{t_1 \le t_2 \dots \le t_{2k}\}$. Now, consider two intervals I, I' such that $\ell_I, \ell_{I'} \in]t_i, t_{i+1}]$. Then I and I' have the same intersection under S. Indeed, if it is not the case, and if we assume that $\ell_I < \ell_{I'}$, there must be an interval I_0 of S such that $\ell_I \le r_{I_0} < \ell_{I'}$ (I_0 intersects I but not I') or such that $r_I \le \ell_{I_0} < r_{I'}$ (I_0 intersects I' but not I). But in both cases we have either $\ell_{I_0} + 1 \in]t_i, t_{i+1}[$ or $\ell_{I_0} - 1 \in]t_i, t_{i+1}[$, a contradiction.

So, we must have at most one interval beginning in each period $]t_i, t_{i+1}]$. It is not possible to have an interval beginning before t_1 or after t_{i+1} because S is also a dominating set. Hence, there at most $2k-1$ intervals in G, and we are done.

By similar arguments, if S is a locating–dominating set, we obtain that there are at most $2k-1$ vertices in $V(G) \setminus S$, hence in total at most $3k-1$ intervals. $\square$

Proposition 10. *The bounds of Theorem 9 are tight for every $k \geq 1$.*

Proof. The bound for identifying codes is reached by odd paths P_{2k-1}. Ordering its intervals $I_1, \ldots, I_{2k-1}$, the set $S = \{I_i \mid i = 1 \bmod 2\}$ is an identifying code. For open locating–dominating sets, consider a path P_{3k-1} whose intervals are ordered $I_1, \ldots, I_{3k-1}$; let $S = \{I_i \mid i = 1, 2 \bmod 3\}$ and add k additional intervals $J_1, \ldots, J_k$, where each J_i is adjacent only to I_{3i-2} and I_{3i-1}. It is easy to check that the resulting graph is a unit interval graph on $4k-1$ vertices. Then S is an open locating–dominating set. For locating–dominating sets, consider the odd path P_{2k-1} and the set S defined for identifying codes, and add to this graph a copy of each interval of S. $\square$

We also obtain the following bound for the order of a unit interval graph with a given metric dimension and a diameter.

Theorem 11. *Let G be a connected unit interval graph on n vertices, of diameter D and with a resolving set of size k. Then $n \leq k(D+2) - 2$.*

Proof. The proof is similar to the one of Theorem 7, except that now the right endpoint of an interval is determined by its left endpoint. Let $s_1, \ldots, s_k$ be the elements of a resolving set S of size k. For each i in $\{1, \ldots, k\}$, ℓ_i is the left endpoint of s_i, and $r_i = \ell_i + 1$ is its right endpoint. Define an ordered set $L^i = \{x^i_1 > x^i_2 > \ldots > x^i_s\}$, where $x^i_1 = \ell_i$ (for $1 < j \leq s$), x^i_{j+1} is the leftmost endpoint of an interval stopping strictly after x^i_j and $x^i_s = x^i_{s+1}$. Similarly, $R^i = \{y^i_1 > y^i_2 > \ldots > y^i_{s'}\}$, with $y^i_1 = r_i$, for $1 < j \leq s'$, y^i_{j+1} is the rightmost endpoint of an interval starting strictly before y^i_j and $y^i_{s'} = y^i_{s'+1}$. In this way, the distance of an interval I to s_i is determined by the position of the right endpoint of I among the points of L^i and the left endpoint of I among the points of R^i. Since the intervals have unit length, the position of the left endpoint of I in R^i is determined by the position of the right endpoint of I in $R^i + 1$ (where $R^i + 1$ denotes the set $\{x + 1 | x \in R^i\}$). Therefore the distance of an interval I to s_i is determined by the position of the right endpoint of I among $L_i \cup (R^i + 1)$.

The distance between the leftmost and the rightmost neighbor of s_i is at least $s + s' - 3$. Therefore, we have $|L^i \cup (R^i + 1)| \leq D + 3$. However, for any i, i' the leftmost point of L^i and $L^{i'}$ are equal, as well as the rightmost point of R^i and of $R^{i'}$. Hence, in total, the union of all sets L^i and $R^i + 1$ contains at most $kD + k + 2$ points, and the distance of an interval in $V(G) \setminus S$ to elements of S is determined by its position compared to the ordering of these points. Moreover, no interval can end before the two first points or after the two last points of $R^i + 1$, so in total there are at most $kD + k - 2$ possibilities. Hence $n \leq kD + k - 2 + k = k(D+2) - 2$. $\square$

Next, we show that the bound of Theorem 11 is almost tight.

Proposition 12. *For every $k \geq 1$ and $D \geq 1$, there exists a unit interval graph of diameter D, a resolving set of size k, and $kD + 1$ vertices.*

Proof. For any $k, D \geq 1$ and $n = kD$, consider the k-th distance-power P^k_{kD+1} of a path on $kD + 1$ vertices (that is, two vertices are adjacent if and only if their distance is at most k in the path P_{kD}). This graph is a unit interval graph of diameter D. Let $\{v_0, \ldots, v_{kD}\}$ be its vertices, ordered according the natural order of the path. Then, the set $S = \{v_0, \ldots, v_{k-1}\}$ forms a resolving set. Indeed, for every i, j with $1 \leq i \leq D-1$ and $0 \leq j \leq k-1$, vertex v_{ik+j} is the unique vertex at distance $i+1$ from all vertices in $\{v_0, \ldots, v_{j-1}\}$ (if $j > 0$) and at distance i from all vertices in $\{v_j, \ldots, v_{k-1}\}$ and vertex v_{kD} is at distance D from all the vertices of S. $\square$

4. Permutation graphs

We now give bounds for permutation graphs.

Theorem 13. *Let G be a permutation graph on n vertices and let S be a k-subset of $V(G)$ with $k \geq 3$. If S is an open locating–dominating set or an identifying code of G, then $n \leq k^2 - 2$. If S is a locating–dominating set of G, then $n \leq k^2 + k - 2$. Hence, $\gamma^{ID}(G) \geq \sqrt{n+2}$, $\gamma^{OLD}(G) \geq \sqrt{n+2}$ and $\gamma^{LD}(G) \geq \sqrt{n + \frac{9}{4}} - \frac{1}{2}$.*

Proof. Let S be a set of k vertices of G. Consider a permutation diagram of G, where each vertex v is represented by two integers: the top index $t(v)$ and the bottom index $b(v)$ of its segment in the diagram. Without loss of generality we can assume that all top indices and all bottom indices are distinct. Let $\{t_1, \ldots, t_k\}$ and $\{b_1, \ldots, b_k\}$ be the two ordered sets of the top and bottom indices of vertices in S. Now, for $1 \le i \le k-1$, let $\mathcal{T}_i$ be the set of top indices (of a vertex of G) that are strictly between t_i and t_{i+1} in the permutation diagram, and let $\mathcal{T}_0, \mathcal{T}_k$ be, respectively, the sets of top indices that are strictly before t_1 and strictly after t_k. For $0 \le i \le k$, let $\mathcal{B}_i$ be the similarly defined set of bottom indices. Observe that every vertex v in $V(G) \setminus S$ has its top and bottom indices $t(v)$ and $b(v)$ in some set $\mathcal{T}_i$ and $\mathcal{B}_j$, respectively.

Now, observe that the segments of two vertices v, w with both $t(v), t(w)$ in some set $\mathcal{T}_i$ and both $b(v), b(w)$ in some set $\mathcal{B}_j$ (hence both v, w belong to $V(G) \setminus S$) intersect exactly the same set of segments of S. Hence if S is an (open) locating–dominating set or an identifying code of G, we must have $v = w$. In other words, each vertex of $V(G) \setminus S$ is uniquely determined by the couple of intervals $(\mathcal{T}_i, \mathcal{B}_j)$ to which its top and bottom indices belong to. We call such a couple a *configuration*.

Further, let $x \in S$, $t(x) = t_i$ and $b(x) = b_j$. Then each of the two potential vertices corresponding to the two configurations $A_1(x) = (\mathcal{T}_{i-1}, \mathcal{B}_{j-1})$ and $A_2(x) = (\mathcal{T}_i, \mathcal{B}_j)$ are intersecting the same subset of S, that is the open neighborhood of x, $N(x) \cap S$. Hence, if S is an open locating–dominating set, then any vertex has neither configuration $A_1(x)$ nor configuration $A_2(x)$ (otherwise this vertex and x would not be totally separated). Also, if S is a locating–dominating set or an identifying code, at most one of $A_1(x)$ and $A_2(x)$ is realized. Note that, by definition, for each pair of distinct vertices $x, y \in S$, we have $A_1(x)$ and $A_1(y)$ are distinct (the same holds for $A_2(x)$ and $A_2(y)$). However we might have $A_1(x) = A_2(y)$ for some $x \neq y$. Nevertheless, if S is an open locating–dominating set, then necessarily $A_1(x) \neq A_2(y)$, since otherwise x, y are not separated. If S is a locating–dominating set or an identifying code, we claim that at least k configurations of the form $A_i(x)$ for $i \in \{1, 2\}$ are not realized. If all of them are distinct, we are done by the previous discussion. Otherwise, consider a maximal sequence $x_1, \dots, x_\ell$ of vertices of S such that $A_2(x_i) = A_1(x_{i+1})$ for every $1 \le i \le \ell - 1$. Then, these vertices form $\ell + 1$ distinct configurations of the form $A_i(x)$ for $i \in \{1, 2\}$. Then, at most one such configuration can be realized, otherwise at least two corresponding vertices would be dominated by the same set of vertices of S. Repeating this argument for all such maximal sequences yields our claim.

Similarly, the two potential vertices corresponding to configurations $A_3(x) = (\mathcal{T}_{i-1}, \mathcal{B}_j)$ and $A_4(x) = (\mathcal{T}_i, \mathcal{B}_{j-1})$ are intersecting in S exactly the closed neighborhood of x, $N[x] \cap S$. Hence, in an identifying code, no vertex has configuration $A_3(x)$ or $A_4(x)$. In an (open) locating–dominating set, at most one of $A_3(x)$ or $A_4(x)$ is realized. Note that for all distinct x, y in S, $A_3(x) \neq A_3(y)$ and $A_4(x) \neq A_4(y)$. For identifying codes, $A_3(x) \neq A_4(y)$, otherwise x and y would not be separated. For (open) locating–dominating sets, considering again a maximal sequence of vertices of S pairwise sharing a configuration and using the same arguments as in the previous paragraph, we get that at least k configurations of the type $A_i(x)$ for $i = 3, 4$ are not realized.

Finally, it is clear that for any two distinct vertices x, y of S, $A_i(x) \neq A_j(y)$ for $i = 1, 2$ and $j = 3, 4$.

We can now proceed with counting the maximum number of realized configurations. There are $(k+1)^2$ configurations in total.

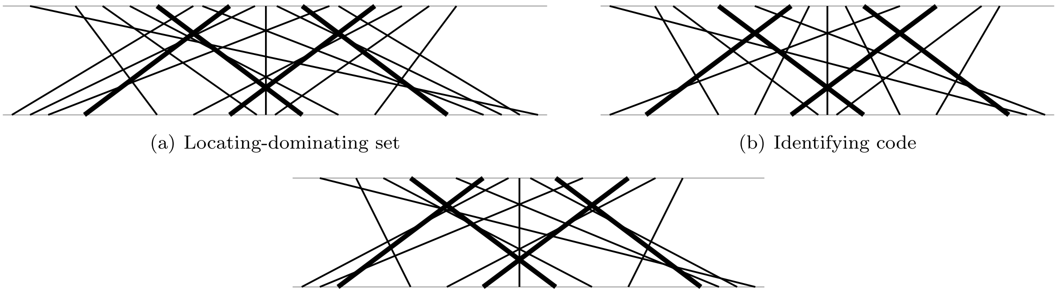
First of all, note that the configuration $(\mathcal{T}_0, \mathcal{B}_0)$ is not realized (otherwise the corresponding vertex would not be dominated). For the same reason, $(\mathcal{T}_k, \mathcal{B}_k)$ is not realized either. Moreover, $(\mathcal{T}_0, \mathcal{B}_0)$ is not a configuration of the form $A_i(x)$ for $i \neq 1$, and $(\mathcal{T}_k, \mathcal{B}_k)$ is not of the form $A_i(x)$ for $i \neq 2$. If $(\mathcal{T}_0, \mathcal{B}_0) = A_1(x)$ for some $x \in S$, then x is only dominated by itself in S. Hence, if S is an open locating–dominating set, this cannot happen. If S is an identifying code or a locating–dominating set, consider once again the maximal sequence $x_1, \dots, x_\ell$ of vertices of S with $x = x_1$ and $A_2(x_i) = A_1(x_{i+1})$ for every $1 \le i \le \ell - 1$. Then, none of the $\ell + 1$ configurations of the type $A_1(x_i)$ or $A_2(x_i)$ can be realized, since none of the corresponding vertices would be dominated. The same argument holds if $(\mathcal{T}_k, \mathcal{B}_k) = A_2(x')$ for some $x' \in S$. Moreover, if the two “saved” configurations are actually the same — i.e. if $x = x_1$ and $x' = x_\ell$ (then S is an independent set) — it is easy to see that there are many additional non-realized configurations. Hence we can assume that $(\mathcal{T}_0, \mathcal{B}_0)$ and $(\mathcal{T}_k, \mathcal{B}_k)$ account for two additional non-realized configurations.

Now, consider the two configurations $(\mathcal{T}_0, \mathcal{B}_k)$ and $(\mathcal{T}_k, \mathcal{B}_0)$: they are both intersecting the whole set of segments and cannot both appear, otherwise the two corresponding vertices are not separated. If one of them is equal to $A_i(x)$ for some $x \in S$ it must be $A_3(x)$ or $A_4(x)$. Then the segment of x is also intersecting all the segments in S. Hence, if S is an identifying code, none of the two configurations $(\mathcal{T}_0, \mathcal{B}_k)$ and $(\mathcal{T}_k, \mathcal{B}_0)$ is realized. However, in that case, we cannot have $(\mathcal{T}_0, \mathcal{B}_k) = A_3(x)$ and $(\mathcal{T}_k, \mathcal{B}_0) = A_4(y)$, otherwise x, y would not be separated. Hence, among $(\mathcal{T}_0, \mathcal{B}_k), (\mathcal{T}_k, \mathcal{B}_0)$ there is at least one non-realized configuration that was not yet counted. If S is an (open) locating–dominating set, then considering a maximal sequence of vertices of S pairwise sharing a configuration and using similar arguments as in the previous paragraph, one can show that again at least one additional configuration is not realized.

If S is an identifying code, none of the $2k$ distinct configurations of the form $A_3(x)$, $A_4(x)$ is realized. Moreover we also proved that at least k configurations of type $A_i(x)$ for $i \in \{1, 2\}$ are not realized, and we also exhibited three additional non-realized configurations. To summarize, we have $(k+1)^2$ configurations, from which $3k + 3$ configurations are not realized, leading to $|V(G) \setminus S| \le k^2 - k - 2$ and so $|V(G)| \le k^2 - 2$.

The same counting gives $|V(G)| \le k^2 - 2$ if S is an open locating–dominating set, and $|V(G)| \le k^2 + k - 2$ if S is a locating–dominating set. $\square$

Proposition 14. *The bounds of Theorem 13 are tight for every $k \ge 1$.*



(a) Locating-dominating set

(b) Identifying code

(c) Open locating-dominating set

Fig. 3. Examples of the constructions described in Proposition 14 that reach the bounds in Theorem 13, with $k = 4$. The bold segments are part of the solution.

Proof. Given k, one can attain the bounds using a solution S inducing a path on k vertices, and realizing all the configurations that were allowed in the proof of Theorem 13. The key observation here is that all configurations of type A_3 or A_4 are distinct, and all configurations of type A_1 or A_2 are distinct too. See Fig. 3 for an illustration. $\square$

Once again, we are able to give a bound on the metric dimension of a permutation graph in terms of its order and diameter, using ideas similar to the ones of Theorem 7.

Theorem 15. *Let G be a connected permutation graph on n vertices, of diameter D and a resolving set of size k. Then $n \le 2k^2(D + 3) + 3k = \Theta(Dk^2)$.*

Proof. As before, consider a permutation diagram of G with top line T and bottom line B, where each vertex v is represented by the top index $t(v) \in T$ and the bottom index $b(v) \in B$ of its segment in the diagram. Let $s_1, \dots, s_k$ be the elements of a resolving set S of G of size k. For every i in $\{1, \dots, k\}$, we define four ordered sets LB^i, LT^i, RB^i, RT^i as follows. Sets LB^i and RB^i contain points of B that are smaller than $b(s_i)$ and greater than $b(s_i)$, respectively; while sets LT^i and RT^i contain points of T that are smaller than $t(s_i)$ and greater than $t(s_i)$, respectively. More precisely, we let $LB^i = \{lb_0^i = b(s_i), lb_1^i, \dots, lb_p^i\}$, where for every $j \in \{1, \dots, p\}$, $lb_j^i = \min\{b(v) \mid d(v, s_i) = j \text{ and } b(v) < b(s_i)\}$, that is lb_j^i is the smallest bottom index of the segment of a vertex v with $d(v, s_i) = j$ and $b(v) < b(s_i)$.[6] Similarly, $RB^i = \{rb_0^i = b(s_i), rb_1^i, \dots, rb_q^i\}$, where for every $j \in \{1, \dots, q\}$, $rb_j^i = \max\{b(v) \mid d(v, s_i) = j \text{ and } b(v) > b(s_i)\}$. Also, $LT^i = \{lt_0^i = t(s_i), lt_1^i, \dots, lt_r^i\}$, where for every $j \in \{1, \dots, r\}$, $lt_j^i = \min\{t(v) \mid d(v, s_i) = j \text{ and } t(v) < t(s_i)\}$. Finally, $RT^i = \{rt_0^i = t(s_i), rt_1^i, \dots, rt_s^i\}$, where for every $j \in \{1, \dots, s\}$, $rt_j^i = \max\{t(v) \mid d(v, s_i) = j \text{ and } t(v) > t(s_i)\}$.

Next, we show that the distance of any vertex v of $V(G) \setminus S$ to s_i is determined by the position of $t(v)$ in LT^i and RT^i, and the position of $b(v)$ in LB^i and RB^i. If $t(v) > t(s_i)$ and $b(v) < b(s_i)$ or $t(v) < t(s_i)$ and $b(v) > b(s_i)$, then $d(v, s_i) = 1$. Otherwise, assume that v lies completely to the left of s_i: $lt_j^i \le t(v) < lt_{j-1}^i$ and $lb_{j'}^i \le b(v) < lb_{j'-1}^i$ for some j, j' with $1 \le j \le r$ and $1 \le j' \le p$ (the case where it lies to the right of s_i is symmetric). Then, we claim that $d(v, s_i) = \min\{j + 1, j' + 1\}$. If $j \le j'$, by definition of j, the segment of v cannot intersect any segment with distance at most $j - 1$ to s_i, hence $d(v, s_i) \ge j + 1$. However, the segment whose top endpoint is lt_j^i must intersect a segment with distance $j - 1$ to s_i, hence it also crosses the segment of v, and $d(v, s_i) \le j + 1$. If $j' < j$, a similar argument holds.

Now, since G has diameter D, we have $|LB^i \cup RB^i| = p + q + 1 \le D + 3$, and $|LT^i \cup RT^i| = r + s + 1 \le D + 3$. Indeed, consider the shortest path P_l of length p starting from the vertex whose bottom index is lb_p and goes to s_i. Consider a similar shortest path P_r of length q from s_i to the vertex whose bottom index is rb_q. If the concatenation of these paths is a shortest path, we are done since in this case $p + q \le D$. Otherwise, notice that a shortcut can only exist around s_i. In fact, it could only be that the penultimate vertex of P_l and the second vertex of P_r are adjacent, or that the ante-penultimate of P_l and the third vertex of P_r are both adjacent to a neighbor of s_i. In any case, the resulting shortest path has length at least $p + q - 2$ and at most D, hence $p + q + 1 \le D + 3$, as claimed.

It follows that using the union of all sets LT^i and RT^i (respectively, LB^i and RB^i), $1 \le i \le k$, induces a partition $\mathcal{P}(T)$ of the line T (respectively, $\mathcal{P}(B)$ of the line B) into at most $k(D + 3) + 1$ parts. Moreover, for any vertex v in $V(G) \setminus S$, the membership of $b(v)$ in a given part of $\mathcal{P}(B)$ and of $t(v)$ in a given part of $\mathcal{P}(T)$ completely determines the distances of v to the vertices in S. Let v be a vertex of $V(G) \setminus S$, with $b(v)$ belonging to some part P of the partition $\mathcal{P}(B)$. For a given i ($1 \le i \le k$), by definition of LT^i and RT^i, there are only two possibilities for the points of $LT^i \cup RT^i$ that $t(v)$ lies between. Hence, there are at most $2k$ vertices in $V(G) \setminus S$ whose associated segment has its bottom point within part P of $\mathcal{P}(B)$. Hence, in total we have

[6] Note that lb_1^i might not exist. In that case we just start the set LB^i from lb_2^i.

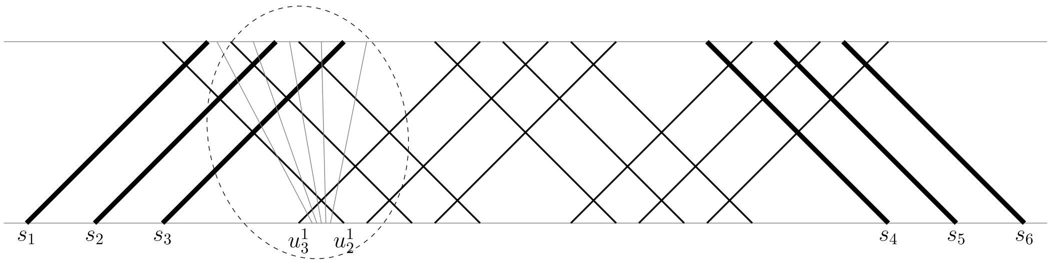


Fig. 4. An example of the construction of Proposition 16 with diameter $D = 6$, a resolving set of size $k = 6$, and order $\Theta(Dk^2)$. Segments similar to the ones in the dashed ellipse are added between the bottom points of every two consecutive segments u^i_{2j} and u^i_{2j+1} of path P_i, for $1 \le i \le k/2$ and $1 \le j \le D/2$. The bold segments $\{s_1, \ldots, s_6\}$ form a resolving set.

$$|V(G)| \le (k(D+3)+1) \cdot 2k + k$$
$$= 2k^2(D+3) + 3k,$$

which completes the proof. □

We do not know if the bound of Theorem 15 is tight, but we are able to provide a construction similar to the one for interval graphs of Proposition 8, showing that this bound is almost tight.

Proposition 16. *For every even $k \ge 2$ and every $D \ge 2$, there are permutation graphs of diameter D, a resolving set of size k and on $\Theta(Dk^2)$ vertices.*

Proof. Consider $k/2$ paths $P_1, \ldots, P_{k/2}$ of length $D-1$ where the path P_{i+1} is obtained from path P_i by a translation (see Fig. 4). Let $P_i = \{u^i_1, \ldots, u^i_D\}$. The endpoints of the paths (i.e. the vertices u^i_1 and u^i_D) form a resolving set. One can add $k/2+2$ vertices that have the bottom index lying between the bottom points of two consecutive segments u^i_{2k} and u^i_{2k+1} of the same path (see the figure). In this way, we add $k/2+2$ segments for each of the $D/2$ intersections of the $k/2$ paths. In the end, the graph has $\Theta(Dk^2)$ vertices. □

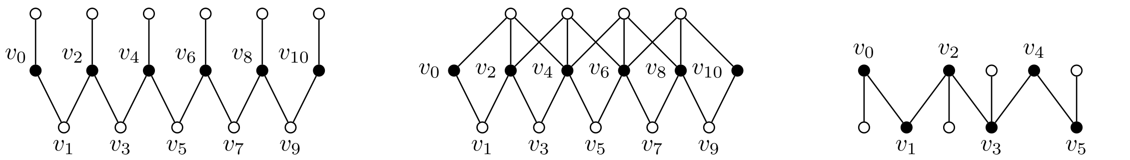
5. Bipartite permutation graphs

A graph is a bipartite permutation graph if it is a permutation graph and it is bipartite. A characterization due to Spinrad [36] uses orderings of the vertices, as follows. Let G be a bipartite graph with parts A and B. An ordering $<$ of A has the *adjacency property* if, for every vertex $b \in B$, its neighborhood $N(b)$ consists of vertices that are consecutive in $<$. It has the *enclosure property* if, for every pair b, b' of vertices in B with $N(b) \subseteq N(b')$, the vertices of $N(b') \setminus N(b)$ are consecutive in $<$. A bipartite graph G with parts A and B is a bipartite permutation graph if and only if it admits an ordering of A that has the adjacency and enclosure properties.

Theorem 17. *Let G be a bipartite permutation graph on n vertices and let S be a k-subset of $V(G)$. If S is a locating–dominating set or an identifying code, then $n \le 3k+2$. If S is an open locating–dominating set, then $n \le 2k+2$. Hence, $\gamma^{\mathrm{ID}}(G) \ge \frac{n-2}{3}$, $\gamma^{OLD}(G) \ge \frac{n-2}{2}$, and $\gamma^{LD}(G) \ge \frac{n-2}{3}$.*

Proof. Let G be a bipartite permutation graph with parts A and B. We may assume that $|A| \ge 2$ and $|B| \ge 2$ (otherwise the graph is a star with isolated vertices and the bounds hold, indeed $\gamma^{\mathrm{ID}}(K_{1,k}) = \gamma^{\mathrm{LD}}(K_{1,k}) = k$; for $k \ge 2$, $K_{1,k}$ has no open locating–dominating set, and $\gamma^{\mathrm{OLD}}(K_{1,1}) = 2$).

Let $<_A$ be an ordering of A that has the adjacency and enclosure properties. We also order the vertices of B using the natural ordering $<_B$ of their neighborhoods within A along $<_A$: given $b_1, b_2 \in B$ with x_1, y_1 and x_2, y_2 the smallest and largest (according to $<_A$) members of $N(b_1)$ and $N(b_2)$, respectively, we have $b_1 <_B b_2$ if $x_1 <_A x_2$ or $x_1 = x_2$ and $y_1 \le y_2$. Note that $<_B$ has the adjacency and the enclosure properties with respect to $<_A$. Indeed, let $a \in A$ and consider its smallest and largest neighbors b_1 and b_2 (then $b_1 \le_B b_2$). Let b be an element between b_1 and b_2. Let x_b (respectively y_b) be the smallest (resp. largest) neighbor in A of b. Since a is adjacent to b_2 and $b <_B b_2$, we have $x_b \le_A a$. Similarly $a \le_A y_b$. Since $<_A$ has the adjacency property, a is in the neighborhood of b and the neighborhood of a consists of vertices that are consecutive in $<_B$ and $<_B$ has the adjacency property. Consider now two vertices a and a' of A with $N(a) \subseteq N(a')$. Without loss of generality, we can assume that $a' < a$. Assume there exists a vertex b in $N(a') \setminus N(a)$ that is larger than all the vertices of $N(a)$. Let b' be an element of $N(a)$. We have $b' <_B b$. Let x_b and $x_{b'}$ be the smallest elements of $N(b)$ and $N(b')$ respectively, and y_b and $y_{b'}$ be the largest elements of $N(b)$ and $N(b')$ respectively. Then $y_b <_A a \le_A y_{b'}$. Since $b' <_B b$, we must have $x_{b'} <_A x_b$ and we get a contradiction because $<_A$ has the enclosure property, $N(b) \subseteq N(b')$ but x_2



(a) Locating-dominating set (b) Identifying code (c) Open locating-dominating set

Fig. 5. Extremal constructions of Proposition 18 with $k = 6$. Black vertices belong to the solution.

and a are in $N(b') \setminus N(b)$ with some elements of $N(b)$ in between. Hence $<_B$ has the enclosure property. Note the order induced on A by $<_B$ (as we defined $<_B$ from $<_A$) is exactly $<_A$.

Consider the set $\mathcal{P}_B$ of all pairs of consecutive vertices in B with respect to $<_B$. We have $|\mathcal{P}_B| = |B| - 1$. Let S be an (open) locating–dominating set or an identifying code. We claim that a vertex a of $A \cap S$ can separate at most two pairs in $\mathcal{P}_B$. Indeed, a pair b, b' in $\mathcal{P}_B$ is separated by a if and only if a is adjacent to exactly one of b, b'. Let the vertices of B be ordered $b_1 <_B \ldots <_B b_{|B|}$. Then, by definition of $<_A$ and $<_B$, there are indices $1 \le \ell, r \le |B|$ with a, b_i non-adjacent if $i \le \ell$ or $i \ge r$, and a, b_i adjacent if $\ell < i < r$. The claim follows.

Now, let $b = b_i$ belong to $B \cap S$. Then b may only separate two pairs in $\mathcal{P}_B$: the ones b itself belongs to, i.e., $\{b_{i-1}, b_i\}$ and $\{b_i, b_{i+1}\}$. However, we claim that the vertices of $S_B = S \cap B$ account only for at most $|S_B|$ pairs in S_B that are *only* separated by vertices of S_B. Indeed, let $b_\ell <_B \ldots <_B b_r$ be a maximal sequence of vertices of B that belong to S_B. Then these $r - \ell + 1$ vertices separate altogether at most $r - \ell + 2$ pairs of $\mathcal{P}_B$. If they separate exactly $r - \ell + 2$ such pairs, then $\ell > 1$ and $r < |B|$. But the pair $\{b_{\ell-1}, b_{r+1}\}$ is also separated by S, that is, by a vertex a in $A \cap S$. But then one of the pairs of $\mathcal{P}_B$ separated by $b_\ell, \ldots, b_r$ is also separated by a. Hence there are at most $r - \ell + 1$ pairs of $\mathcal{P}_B$ separated by these $r - \ell + 1$ vertices. Repeating this counting for all such maximal subsequences yields the claim.

Moreover, observe that if S is an open locating–dominating set, a vertex b in $S \cap B$ cannot separate any pair in $\mathcal{P}_B$.

We can use the same arguments reversing the role of $<_B$ and $<_A$ and the set $\mathcal{P}_A$ of pairs of consecutive vertices of A.

To summarize, if S is a locating–dominating set or an identifying code, the vertices in $S \cap A$ may separate at most $2|S \cap A|$ pairs of $\mathcal{P}_B$ and separate at most $|S \cap A|$ pairs of $\mathcal{P}_A$ that are not separated by a vertex of $S \cap B$, and vice-versa the vertices in $S \cap B$ may separate at most $2|S \cap B|$ pairs of $\mathcal{P}_A$ and separate at most $|S \cap B|$ pairs of $\mathcal{P}_B$ that are not separated by a vertex of $S \cap A$. Hence $|B| - 1 = |\mathcal{P}_B| \le 2|S \cap A| + |S \cap B|$ and $|A| - 1 = |\mathcal{P}_A| \le 2|S \cap B| + |S \cap A|$. In total, $n = |A| + |B| \le 3|S| + 2$ and we are done.

Similarly, if S is an open locating–dominating set, we have $|B| - 1 = |\mathcal{P}_B| \le 2|S \cap A|$ and $|A| - 1 = |\mathcal{P}_A| \le 2|S \cap B|$, yielding $n \le 2|S| + 2$. □

The bounds of Theorem 17 are almost tight.

Proposition 18. *For every $k \ge 1$, there exist three bipartite permutation graphs with a locating–dominating set, an identifying code and an open locating–dominating set of size $3k - 1$, $3k - 3$ and $2k - 2$, respectively.*

Proof. For location–domination, consider an odd path P_{2k-1} with $V(P_{2k-1}) = \{v_0, \ldots, v_{2k-2}\}$, let $S = \{v_i \mid i = 0 \bmod 2\}$ and attach a pendant vertex to every vertex in S. This graph is a bipartite permutation graph (observe that S, together with its natural ordering, has the adjacency and enclosure properties), it has $n = 3k - 1$ vertices and S is a locating–dominating set.

For identifying codes, again select the odd path P_{2k-1} with $S = \{v_i \mid i = 0 \bmod 2\}$, but now for every $i \in \{2, \ldots, 2k - 4\}$ add a vertex adjacent to $\{v_i, v_{i+2}, v_{i+4}\}$. Again S is an identifying code and the graph has $n = 3k - 3$ vertices.

For open locating–dominating sets, select any path P_k with $S = V(P_k) = \{v_0, \ldots, v_{k-1}\}$, and attach a pendant vertex to every vertex in $S \setminus \{v_1, v_{k-2}\}$. Again S is an open locating–dominating set and the graph has $n = 2k - 2$ vertices.

The constructions are illustrated in Fig. 5. □

Theorem 19. *Let G be a connected bipartite permutation graph on n vertices, of diameter D and a resolving set of size k. Then $n \le k(2D - 1) + 2 = \Theta(Dk)$.*

Proof. Let A and B be the two parts of the bipartition of G, and consider two orderings $<_A$ of A and $<_B$ of B that have the adjacency and enclosure properties. Let $S = \{s_1, \ldots, s_k\}$ be a resolving set of G, and assume without loss of generality that for some $i \in \{1, \ldots, k\}$, $s_i \in A$. Then, the sets A and B can be partitioned into parts consisting of consecutive vertices (with respect to $<_A$ and $<_B$), where the vertices in each part have the same distance to s_i. Moreover, the vertices in A have even distances to s_i, while the vertices in B have odd distances to s_i. The number of parts in A and in B is at most D.

Repeating this process for each vertex of S, we have partitioned the vertices in $A \setminus S$ and of $B \setminus S$ into at most $k(D-1)+1$ parts each. Each part may contain at most one vertex of $V(G) \setminus S$ (since the membership in a part determines the distances to the vertices of S). Hence, we have

$$|V(G)| \le 2(k(D-1)+1)+k$$
$$= k(2D-1)+2. \qquad \square$$

Next, we show that Theorem 19 is asymptotically tight.

Proposition 20. *For every even $k \ge 2$ and every $D \ge 2$, there exists a bipartite permutation graph of diameter D, a resolving set of size k and $\Theta(Dk)$ vertices.*

Proof. Simply consider the construction of Proposition 16 built from $k/2$ paths of length $D-1$ each (omitting the second part of the construction of Proposition 16). This bipartite permutation graph has $Dk/2$ vertices and a resolving set of size k. $\square$

6. Algorithm and bounds for cographs

The *cotree* of a cograph G is a tree where the leaves are the vertices of G, and the inner nodes are of type $\oplus$ and $\bowtie$. This tree represents the construction of G using the two operations. A cograph can be recognized in linear time and its corresponding cotree can be constructed in linear time too [23]. Many problems can be computed in linear time in cographs using their cotree representation and by simple bottom-up computation. Epstein, Levin and Woeginger [17] gave such an algorithm for computing the metric dimension. Observe that connected cographs have diameter at most 2, hence, as already discussed, for any connected cograph G we have $dim(G) \le \gamma^{\text{LD}}(G) \le dim(G)+1$, and $\gamma^{\text{LD}}(G) = dim(G)+1$ if and only if every minimum resolving set has a non-dominated vertex. The latter fact is computed by the algorithm of [17], which can therefore be used for computing the location–domination number of a cograph.

In this subsection, we will give a similar linear-time algorithm for Identifying Code, and we will give linear lower bounds on the value of parameters γ^{LD}, dim, γ^{ID} and γ^{OLD} in terms of the order.

6.1. The algorithm

We describe in detail the algorithm for identifying codes (the one for open locating–dominating sets is very similar). We denote by $sep_{ID}(G)$ the smallest size of a *separating set*, that is, a set $S \subseteq V(G)$ where for every pair u, v of distinct vertices, there is an element of S dominating exactly one of u, v (it is an identifying code without the condition of being a dominating set). It follows from the definitions that $sep_{ID}(G) \le \gamma^{\text{ID}}(G) \le sep_{ID}(G)+1$, where the upper bound is reached if and only if for every smallest separating set there is a non-dominated vertex. Therefore, if we can compute $sep_{ID}(G)$ as well as decide the latter fact, then we can compute $\gamma^{\text{ID}}(G)$.

We define ID-EMP(G) to be the property that for a graph G, every minimum separating set leaves a non-dominated vertex of G; ID-UNIV(G) is the property that for every minimum separating set S of G, there exists a vertex of G that is dominated by all vertices of S.

The advantage of using $sep_{ID}(G)$ comes from the following lemma.

Lemma 21. *Let G_1, G_2 be two twin-free graphs with $sep_{ID}(G_1) = k_1$ and $sep_{ID}(G_2) = k_2$. Then, $k_1 + k_2 \le sep_{ID}(G_1 \oplus G_2) \le k_1 + k_2 + 1$, where the upper bound is reached if and only if properties ID-EMP(G_1) and ID-EMP(G_2) hold. Moreover, suppose $G_1 \bowtie G_2$ is a twin-free graph, then $k_1 + k_2 \le sep_{ID}(G_1 \bowtie G_2) \le k_1 + k_2 + 1$ and the upper bound is reached if and only if properties ID-UNIV(G_1) and ID-UNIV(G_2) hold.*

Proof. Note that in both $G_1 \oplus G_2$ and $G_1 \bowtie G_2$, a vertex in G_1 cannot separate a pair in G_2, and vice-versa. Hence, for every separating set of $G_1 \oplus G_2$ or $G_1 \bowtie G_2$, its restriction to G_i for $i \in \{1, 2\}$ is a separating set of G_i. This proves the two lower bounds.

For the upper bounds, let S_1 and S_2 be minimum separating sets of G_1 and G_2, respectively. If $S = S_1 \cup S_2$ is not a separating set of $G_1 \oplus G_2$ or $G_1 \bowtie G_2$, by the previous observation, there must be a pair u, v with $u \in G_1$ and $v \in G_2$ that is not separated. In the case of $G_1 \oplus G_2$, these two vertices must both be non-dominated by S, and this is the only non-separated pair. Then, adding one of them gives a separating set of size k_1+k_2+1. For the case $G_1 \bowtie G_2$, u is dominated by all vertices of S_2, and v is dominated by all vertices of S_1. Hence both u, v must be dominated by all vertices of S, and this is the only non-separated pair. Since u, v are not twins, there must be a vertex w that separates them; $S \cup \{w\}$ is a separating set of $G_1 \bowtie G_2$ of size $k_1 + k_2 + 1$. $\square$

Using the following lemma, it is easy to keep track of the properties ID-EMP and ID-UNIV while parsing the cotree structure of a cograph G.

Lemma 22. *We have:*

1. *ID-EMP(K_1) and ID-UNIV(K_1);*
2. *ID-EMP($G_1 \oplus G_2$) if and only if ID-EMP(G_1) or ID-EMP(G_2);*

3. *ID-UNIV*$(G_1 \oplus G_2)$ *if and only if one of* G_1, G_2 *(say* G_1*) is* K_1, *ID-UNIV*(G_2) *and* ¬*ID-EMP*(G_2)*;*
4. *ID-EMP*$(G_1 \bowtie G_2)$ *if and only if* $G_1 = K_1$, ¬*ID-UNIV*(G_2) *and ID-EMP*(G_2)*;*
5. *ID-UNIV*$(G_1 \bowtie G_2)$ *if and only if ID-UNIV*(G_1) *or ID-UNIV*(G_2)*.*
6. *If* ¬*ID-EMP*(G) *and* ¬*ID-UNIV*(G)*, then there exists a minimum separating set* S *of* G *such that every vertex of* G *is dominated by some element of* S *but no vertex of* G *is dominated by the entire set* S*.*

Proof. We prove the lemma by induction using the cotree structure of cographs.

The first item is clearly true. For the second item, assume ID-EMP$(G_1 \oplus G_2)$. By Lemma 21, if ¬ID-EMP(G_1) or ¬ID-EMP(G_2), then any minimum separating set of $G_1 \oplus G_2$ is the union of a minimum separating set of G_1 and one of G_2. Hence if both ¬ID-EMP(G_1) and ¬ID-EMP(G_2), then ¬ID-EMP$(G_1 \oplus G_2)$, which is a contradiction. Now, if ID-EMP(G_1) and ¬ID-EMP(G_2) (or vice-versa), by Lemma 21, we have ID-EMP$(G_1 \oplus G_2)$. If both ID-EMP(G_1) and ID-EMP(G_2), then again by Lemma 21, $sep_{ID}(G_1 \oplus G_2) = sep_{ID}(G_1) + sep_{ID}(G_2) + 1$, but since no vertex of G_1 dominates any vertex of G_2 (and vice-versa), there must remain a non-dominated vertex in $G_1 \oplus G_2$.

For the third item, assume ID-UNIV$(G_1 \oplus G_2)$. If none of G_1, G_2 is K_1, then there must be a code vertex in both G_1, G_2, which would imply that ¬ID-UNIV$(G_1 \oplus G_2)$ and contradict the hypothesis. Thus we may assume $G_1 = K_1$, and let S_2 be a minimum separating set of G_2. By Lemma 21, if ID-EMP(G_2), $sep_{ID}(G_1 \oplus G_2) = sep_{ID}(G_1) + sep_{ID}(G_2) + 1$. But then $S' = S_2 \cup V(K_1)$ is a minimum separating set of $G_1 \oplus G_2$ without a vertex dominated by the whole of S', a contradiction. Hence, ¬ID-EMP(G_2). If we also have ¬ID-UNIV(G_2), by induction hypothesis and using item 6, there exists a minimum separating set S_2 of G_2 with no vertex dominated by the whole set S_2 and with all vertices of G_2 dominated by S_2. Hence S_2 is a minimum separating set of $G_1 \oplus G_2$ without a vertex dominated by the whole set S_2 and we have ¬ID-UNIV$(G_1 \oplus G_2)$, a contradiction. For the converse, if $G_1 = K_1$, ID-UNIV(G_2) and ¬ID-EMP(G_2), then by Lemma 21, $sep_{ID}(G_1 \oplus G_2) = sep_{ID}(G_1) + sep_{ID}(G_2)$, and it is clear that there is no minimum separating set of $G_1 \oplus G_2$ containing the vertex of K_1. Hence every minimum separating set of $G_1 \oplus G_2$ is a minimum separating set of G_2, and since ID-UNIV(G_2), we are done.

For the fourth item, assume that ID-EMP$(G_1 \bowtie G_2)$. Again if none of G_1, G_2 is K_1 there must be a code vertex in each part, a contradiction. Assume $G_1 = K_1$. If ID-UNIV(G_2), by Lemma 21, $sep_{ID}(G_1 \bowtie G_2) = sep_{ID}(G_1) + sep_{ID}(G_2) + 1$. Since ID-EMP$(G_1 \bowtie G_2)$, the vertex of K_1 cannot belong to any minimum separating set. Consider a minimum separating set S_2 of G_2: since ID-UNIV(G_2), there is a vertex x of G_2, which is dominated by the whole set S_2. But since G is twin-free, x has a non-neighbor y in G_2 (and $y \notin S_2$). Then $S_2 \cup \{y\}$ is a (minimum) separating set of $G_1 \bowtie G_2$. Since ID-EMP$(G_1 \bowtie G_2)$, there is a vertex u, necessarily in G_2, that is not dominated by $S_2 \cup \{y\}$. If x is not adjacent to u, we could choose u to be y and $S_2 \cup \{u\}$ would be a (minimum) separating set of $G_1 \bowtie G_2$ without any non-dominated vertex (since S_2 is a separating set for G_2, there is at most one vertex of G_2 having no neighbors in S_2), a contradiction. Hence, x is adjacent to u and $u \neq y$ and y is not adjacent to u and x. But since $u \neq y$ and S_2 is a separating set of G_2, in order to be separated from u, y must be adjacent to some vertex s of S_2. Then, y, s, x, u induce a path on four vertices, a contradiction since we assumed $G_1 \bowtie G_2$ is a cograph. Hence ¬ID-UNIV(G_2). Now, if we also have ¬ID-EMP(G_2), using induction hypothesis and item 6, there is a minimum separating set S_2 of G_2 that dominates each vertex of G_2 and such that no vertex of S_2 is dominated by all the other vertices of S_2. Hence S_2 is also a separating set of G_1 and ¬ID-EMP$(G_1 \bowtie G_2)$, a contradiction. For the converse, assume $G = K_1$, ¬ID-UNIV(G_2) and ID-EMP(G_2). Then by Lemma 21 $sep_{ID}(G_1 \bowtie G_2) = sep_{ID}(G_1) + sep_{ID}(G_2) = sep_{ID}(G_2)$. Let S be a minimum separating set of $G_1 \bowtie G_2$: then $S \setminus V(K_1)$ is a minimum separating set of G_2, hence $S \setminus V(K_1) = S$ and thus we have ID-EMP$(G_1 \bowtie G_2)$ (since ID-EMP(G_2)).

For the fifth item, suppose that ¬ID-UNIV(G_1) and ¬ID-UNIV(G_2). Then, by Lemma 21, $sep_{ID}(G_1 \bowtie G_2) = sep_{ID}(G_1) + sep_{ID}(G_2)$ and the restriction of a separating set S of $G_1 \bowtie G_2$ to G_i ($i \in \{1, 2\}$) is a separating set of G_i. Since none of G_1, G_2 is K_1, there is vertex of S in each part, hence we have ¬ID-UNIV$(G_1 \bowtie G_2)$. For the converse, assume that ID-UNIV(G_1) or ID-UNIV(G_2). If both ID-UNIV(G_1) and ID-UNIV(G_2), by Lemma 21, $sep_{ID}(G_1 \bowtie G_2) = sep_{ID}(G_1) + sep_{ID}(G_2) + 1$. Again, since the restriction of a separating set S of $G_1 \bowtie G_2$ to G_i ($i \in \{1, 2\}$) is a separating set of G_i, a minimum separating set S of $G_1 \bowtie G_2$ consists of one separating set S_1 of G_1, one separating set S_2 of G_2, with an additional vertex in say G_1. Then the vertex in G_2 that is dominated by the whole S_2 is also dominated by the whole set S. The other case is handled similarly.

For the sixth item, we use the previous results. Assume first that $G = G_1 \oplus G_2$ and that ¬ID-EMP(G) and ¬ID-UNIV(G). Then we have in particular, using item 2, ¬ID-EMP(G_1) and ¬ID-EMP(G_2). Consider any minimum separating sets S_1 of G_1 and S_2 of G_2 that dominates all the vertices of G_1 and G_2 respectively. By Lemma 21, $S_1 \cup S_2$ is a minimum separating set of G that dominates all the vertices of G. Since S_1 and S_2 are both non-empty, $S_1 \cup S_2$ has no vertex dominated by all the vertices of $S_1 \cup S_2$. Assume now that $G = G_1 \bowtie G_2$ and that ¬ID-EMP(G) and ¬ID-UNIV(G). Using item 5, we have ¬ID-UNIV(G_1) and ¬ID-UNIV(G_2). Let S_1 (respectively S_2) be a minimum separating set of G_1 (respectively G_2) with no vertex dominated by all the vertices of S_1 (respectively S_2). By Lemma 21, $S_1 \cup S_2$ is a minimum separating set of G and no vertex is dominated by all the vertices of $S_1 \cup S_2$. Moreover, S_1 and S_2 are both non-empty, hence $S_1 \cup S_2$ dominates all the vertices of G. □

Observe that if a cograph is twin-free, then every intermediate cograph obtained during its construction is twin-free too, since operations $\oplus$ and $\bowtie$ preserve twins. This fact together with Lemmas 21 and 22, implies a linear time algorithm which constructs an identifying code of a minimum size for a given cograph (based on parsing of it cotree structure).

Moreover similar ideas lead to an algorithm for open locating–dominating sets, the details of which are left to the reader.

Theorem 23. *There exist linear-time algorithms that construct a minimum identifying code and a minimum open locating–dominating set of a cograph.*

6.2. Bounds for cographs

We now use the previous discussion to give tight lower bounds on the identifying code number, (open) locating–domination number and metric dimension of cographs.

Theorem 24. *Let G be a twin-free cograph on $n \geq 2$ vertices with an identifying code of size k. Then, $n \leq 2k-2$, or equivalently $\gamma^{ID}(G) \geq \frac{n+2}{2}$.*

Proof. In fact, we prove the following stronger facts (for a cograph G on at least two vertices):

1. if $\neg$ID-EMP(G) and $\neg$ID-UNIV(G), then $sep_{ID}(G) \geq \frac{n+2}{2}$;
2. if ID-EMP(G) and $\neg$ID-UNIV(G) or $\neg$ID-EMP(G) and ID-UNIV(G), then $sep_{ID}(G) \geq \frac{n+1}{2}$;
3. if ID-EMP(G) and ID-UNIV(G), then $sep_{ID}(G) \geq \frac{n}{2}$.

The proof uses induction on the order of the cograph and the fact that any cograph is built recursively from two cographs using operation $\oplus$ or $\bowtie$. The claim is clearly true for the only twin-free cograph on two vertices, $\overline{K_2}$, hence assume $n > 2$. We just have to prove the result for $G = G_1 \oplus G_2$ since everything is symmetric by taking the complement and exchanging ID-EMP(G) with ID-UNIV(G).

Assume first that $G_1 = K_1$. Then G_2 has $n_2 \geq 2$ vertices and by induction the properties $1, 2, 3$ hold for G_2. We have ID-EMP(G_1) and so ID-EMP(G). If ID-UNIV(G) holds, then by Lemma 22, we have ID-UNIV(G_2) and $\neg$ID-EMP(G_2), hence $sep_{ID}(G) \geq sep_{ID}(G_2) \geq \frac{n_2+1}{2} = \frac{n}{2}$, and we are done. Assume now that $\neg$ID-UNIV(G). If ID-EMP(G_2), then by Lemma 21, $sep_{ID}(G) = sep_{ID}(G_1) + sep_{ID}(G_2) + 1 \geq \frac{n_2}{2} + 1 \geq \frac{n+2}{2}$ and we are done. Otherwise, we have $\neg$ID-EMP(G_2) and by item 3 of Lemma 22, we also have $\neg$ID-UNIV(G_2), hence $sep_{ID}(G) \geq sep_{ID}(G_2) \geq \frac{n_2+2}{2} \geq \frac{n+1}{2}$.

Assume now that none of G_1, G_2 is K_1, then by induction, the properties hold for G_1 and G_2 and we have $\neg$ID-UNIV(G). If both $\neg$ID-EMP(G_1) and $\neg$ID-EMP(G_2), then we also have $\neg$ID-EMP(G) and $sep_{ID}(G) \geq sep_{ID}(G_1) + sep_{ID}(G_2) \geq \frac{n_1+1}{2} + \frac{n_2+1}{2} \geq \frac{n+2}{2}$ and we are done. If both ID-EMP(G_1) and ID-EMP(G_2), then ID-EMP(G) and $sep_{ID}(G) \geq sep_{ID}(G_1) + sep_{ID}(G_2) + 1 \geq \frac{n_1}{2} + \frac{n_2}{2} + 1 \geq \frac{n+1}{2}$. Finally, if only one property holds, say ID-EMP(G_1), then ID-EMP(G) and $sep_{ID}(G) \geq sep_{ID}(G_1) + sep_{ID}(G_2) \geq \frac{n_1}{2} + \frac{n_2+1}{2} \geq \frac{n+1}{2}$. $\square$

Proposition 25. *The bound of Theorem 24 is tight for infinitely many cographs.*

Proof. We construct, inductively, graphs reaching the bound. Assume there are graphs G_n^1,G_n^2,G_n^3,G_n^4 on n vertices such that

- $sep_{ID}(G_n^1) = \lceil \frac{n+2}{2} \rceil$, $\neg$ID-EMP(G_n^1) and $\neg$ID-UNIV(G_n^1);
- $sep_{ID}(G_n^2) = \lceil \frac{n+1}{2} \rceil$, ID-EMP($G_n^2$) and $\neg$ID-UNIV(G_n^2);
- $sep_{ID}(G_n^3) = \lceil \frac{n+1}{2} \rceil$, $\neg$ID-EMP(G_n^3) and ID-UNIV(G_n^3);
- $sep_{ID}(G_n^4) = \lceil \frac{n}{2} \rceil$, ID-EMP($G_n^4$), ID-UNIV($G_n^4$) and G_n^4 does not have a universal vertex.

Then the graphs $G_{n+1}^1 = \overline{K_3} \bowtie G_{n-2}^2$, $G_{n+1}^2 = K_1 \oplus G_n^4$, $G_{n+1}^3 = K_1 \bowtie G_n^4$, $G_{n+1}^4 = K_1 \oplus G_n^3$, satisfy the properties for $n+1$ vertices.

Starting with $G_3^2 = \overline{K_3}$, $G_3^3 = P_3$, $G_4^2 = \overline{K_4}$ and $G_4^3 = C_4$, we obtain a sequence of graphs G_n^i for $i \geq 2$ and $n \geq 4$ satisfying the properties. We obtain the graphs G_n^1 for $n \geq 6$ using the graphs G_{n-2}^2. $\square$

We can prove similar results for locating–dominating sets. Since the proofs are very similar to that of identifying codes, we defer them to Appendix A.

Theorem 26. *Let G be a connected cograph on $n \geq 2$ vertices, having a resolving set of size k and a locating–dominating set of size d. Then, $n \leq 3k \leq 3d$.*

The bound of Theorem 26 is tight for infinitely many cographs.

Proposition 27. *There exist infinitely many cographs where both inequalities of Theorem 26 are simultaneously achieved.*

7. Conclusion

We conclude with some open problems. It would be interesting to know whether similar bounds, as the ones we established here, hold for other standard graph classes. One specific case that is worth studying is the metric dimension of planar graphs and line graphs. For these two classes, such bounds are known to exist for locating–dominating sets and identifying codes ($O(k)$ for planar graphs [35] and $O(k^2)$ for line graphs [19]). Do bounds of the form $O(Dk)$ and $O(Dk^2)$, respectively, hold for planar graphs and line graphs?[7]

We also remark that during the writing of this paper, the fourth author, together with several colleagues [10], proved that for any graph G of order n and VC-dimension at most d, the bound $n \leq O(k^d)$ holds, where k is the size of an identifying code of G (the same bound also applies to (open) locating–dominating sets). In particular, interval graphs have VC-dimension at most 2, and permutation graphs have VC-dimension at most 3. Hence, the result of [10] generalizes some of our results (but our bounds are more precise).

Appendix A. Locating–dominating sets and metric dimension of cographs

As mentioned in the introduction, if G has diameter 2, we have $dim(G) \leq \gamma^{\mathrm{LD}}(G) \leq dim(G)+1$, where the upper bound is reached if and only if for every smallest resolving set there is a non-dominated vertex. Since we will use the cotree structure of cographs, we have to deal with not connected graphs for which the difference between $dim(G)$ and $\gamma^{\mathrm{LD}}(G)$ can be more than one. As before, we denote by $sep_{LD}(G)$ the smallest size of a *separating set*, that is, a set $S \subseteq V(G)$ that separates all the vertices of $V(G) \setminus S$ (it is a locating–dominating set without the condition of being a dominating set). If G is a connected cograph, it has diameter 2 and a separating set is equivalent to a resolving set, in particular $sep_{LD}(G) = dim(G)$. If G is not connected, then the two parameters can be different since in a resolving set, one vertex per connected component could be not dominated. We define LD-EMP(G) as the property that for a graph G, every minimum separating set leaves a non-dominated vertex; LD-UNIV(G) is the property that every minimum separating set S of G leaves a vertex in $G \setminus S$ that is dominated by all vertices of S. We have $sep_{LD}(G) \leq \gamma^{\mathrm{LD}}(G) \leq sep_{LD}(G)+1$ and $\gamma^{\mathrm{LD}}(G) = sep_{LD}(G)+1$ if and only if LD-EMP(G).

Note that S is a separating set of G if and only if it is a separating set of the complement of G. Moreover, the following hold:

LD-EMP(G) if and only if LD-UNIV($\overline{G}$)
LD-UNIV(G) if and only if LD-EMP($\overline{G}$)

We have the following lemma.

Lemma 28. *Let G_1, G_2 be two cographs with $sep_{LD}(G_1) = k_1$ and $sep_{LD}(G_2) = k_2$. Then, $k_1 + k_2 \leq sep_{LD}(G_1 \oplus G_2) \leq k_1 + k_2 + 1$, where the upper bound is reached if and only if we have LD-EMP(G_1) and LD-EMP(G_2). Moreover, $k_1 + k_2 \leq sep_{LD}(G_1 \bowtie G_2) \leq k_1 + k_2 + 1$ and the upper bound is reached if and only if we have LD-UNIV(G_1) and LD-UNIV(G_2).*

Proof. Note that in both $G_1 \oplus G_2$ and $G_1 \bowtie G_2$, a vertex in G_1 cannot separate a pair in G_2, and vice-versa. Hence, for every separating set of $G_1 \oplus G_2$ or $G_1 \bowtie G_2$, its restriction to G_i for $i \in \{1,2\}$ is a separating set of G_i. This proves the two lower bounds.

For the upper bounds, let S_1 and S_2 be minimum separating sets of G_1 and G_2, respectively. If $S = S_1 \cup S_2$ is not a separating set of $G_1 \oplus G_2$ or $G_1 \bowtie G_2$, by the previous observation, there must be a pair u, v with $u \in G_1$ and $v \in G_2$ that is not separated. In the case of $G_1 \oplus G_2$, these two vertices must both be non-dominated by S, and this is the only non-separated pair. Then, adding one of them gives a separating set of size k_1+k_2+1. For the case $G_1 \bowtie G_2$, u is dominated by all vertices of S_2, and v is dominated by all vertices of S_1. Hence both u, v must be dominated by all vertices of S, and this is the only non-separated pair. Then $S \cup \{u\}$ is a resolving set of $G_1 \bowtie G_2$ of size k_1+k_2+1. □

Using the following lemma, it is easy to keep track of the properties LD-EMP and LD-UNIV while parsing the cotree structure of a cograph G.

Lemma 29. *We have:*

1. *LD-EMP(K_1) and LD-UNIV(K_1);*
2. *LD-EMP($G_1 \oplus G_2$) if and only if LD-EMP(G_1) or LD-EMP(G_2);*

[7] After the completion and submission of this paper, this has been investigated in [4] by two of the authors with other colleagues. The answer turns out to be negative for both planar graphs and line graphs. In that paper, it is shown that there exist line graphs of diameter 4 and order $\Omega(2^k)$, outerplanar graphs of order $\Theta(kD^2)$, and planar graphs of metric dimension 3 and order $\Theta(D^3)$. Nevertheless, for planar graphs, it is proved there that $n = O(D^4k^4)$ holds.

3. *LD-UNIV*($G_1 \oplus G_2$) *if and only if one of* G_1, G_2 *(say* G_1*) is* K_1*, LD-UNIV*(G_2) *and* ¬*LD-EMP*(G_2)*;*
4. *LD-EMP*($G_1 \bowtie G_2$) *if and only if* $G_1 = K_1$*,* ¬*LD-UNIV*(G_2) *and LD-EMP*(G_2)*;*
5. *LD-UNIV*($G_1 \bowtie G_2$) *if and only if LD-UNIV*(G_1) *or LD-UNIV*(G_2)*.*
6. *If* ¬*LD-EMP*(G) *and* ¬*LD-UNIV*(G)*, there is a minimum separating set* S *of* G *such that all the vertices are dominated by some vertex of* S *and there is no vertex out of* S *that dominates all* S*.*

Proof. Since taking the complement is the same as exchanging LD-EMP to LD-UNIV and $\bowtie$ to $\oplus$, we just have to prove items 1, 2, 3 and 6. We prove these items by induction.

The first item is clear. For the second item, assume LD-EMP($G_1 \oplus G_2$). By Lemma 28, if ¬LD-EMP(G_1) or ¬LD-EMP(G_2), then any minimum separating set of $G_1 \oplus G_2$ is the union of a minimum separating set of G_1 and one of G_2. Hence if both ¬LD-EMP(G_1) and ¬LD-EMP(G_2), then ¬LD-EMP($G_1 \oplus G_2$). Now, if LD-EMP(G_1) and ¬LD-EMP(G_2) (or vice-versa), for the same reason we have LD-EMP($G_1 \oplus G_2$). If both LD-EMP(G_1) and LD-EMP(G_2), then again by Lemma 28, $sep_{LD}(G_1 \oplus G_2) = sep_{LD}(G_1) + sep_{LD}(G_2) + 1$, but since no vertex of G_1 dominates any vertex of G_2 (and vice-versa), there must remain a non-dominated vertex.

For the third item, assume LD-UNIV($G_1 \oplus G_2$). If none of G_1, G_2 is K_1, then there must be a code vertex in both G_1, G_2, hence ¬LD-UNIV($G_1 \oplus G_2$). Hence assume $G_1 = K_1$, and let S_2 be a minimum separating set of G_2. By Lemma 28, if LD-EMP(G_2), $sep_{LD}(G_1 \oplus G_2) = sep_{LD}(G_1) + sep_{LD}(G_2) + 1$. But then $S' = S_2 \cup V(K_1)$ is a minimum separating set of $G_1 \oplus G_2$ without a vertex dominated by the whole of S', a contradiction. Hence, ¬LD-EMP(G_2). If ¬LD-UNIV(G_2), using induction and item 6, there is a minimum separating set S_2 of G_2 without any vertex dominated by the whole of S_2 and all the vertices dominated by some vertex of S_2. Then S_2 is a minimum separating set of G without any vertex dominated by the whole set S_2, a contradiction. Hence LD-UNIV(G_2). For the converse, if $G_1 = K_1$, LD-UNIV(G_2) and ¬LD-EMP(G_2), then by Lemma 28, $sep_{LD}(G_1 \oplus G_2) = sep_{LD}(G_1) + sep_{LD}(G_2)$, and it is clear that the vertex of K_1 does not belong to a any minimum separating set of $G_1 \oplus G_2$. Hence every minimum separating set of $G_1 \oplus G_2$ is a minimum separating set of G_2, and we are done.

For the sixth item, we use the previous results. Assume first that $G = G_1 \oplus G_2$ and that ¬LD-EMP(G) and ¬LD-UNIV(G). Then we have in particular, using item 2, ¬LD-EMP(G_1) and ¬LD-EMP(G_2). Consider any minimum separating sets S_1 of G_1 and S_2 of G_2 that dominates all the vertices of G_1 and G_2 respectively. By Lemma 28, $S_1 \cup S_2$ is a minimum separating set of G that dominates all the vertices of G. Since S_1 and S_2 are both non-empty, $S_1 \cup S_2$ has no vertex dominated by all the vertices of $S_1 \cup S_2$. Assume now that $G = G_1 \bowtie G_2$ and that ¬LD-EMP(G) and ¬LD-UNIV(G). Using item 5, we have ¬LD-UNIV(G_1) and ¬LD-UNIV(G_2). Let S_1 (respectively S_2) be a minimum separating set of G_1 (respectively G_2) with no vertex dominated by all the vertices of S_1 (respectively S_2). By Lemma 28, $S_1 \cup S_2$ is a minimum separating set of G and no vertex is dominated by all the vertices of $S_1 \cup S_2$. Moreover, S_1 and S_2 are both non-empty, hence $S_1 \cup S_2$ dominates all the vertices of G. □

We can now prove Theorem 26.

Proof of Theorem 26. In fact, we prove the following stronger facts:

1. if ¬LD-EMP(G) and ¬LD-UNIV(G), $sep_{LD}(G) \geq \frac{n}{3}$;
2. if LD-EMP(G) and ¬LD-UNIV(G) or ¬LD-EMP(G) and LD-UNIV(G), $sep_{LD}(G) \geq \frac{n+1}{3}$;
3. if LD-EMP(G) and LD-UNIV(G), $sep_{LD}(G) \geq \frac{n+2}{3}$.

The claim is clearly true for K_2 and $\overline{K_2}$, hence assume $n > 2$. We just have to prove the result for $G = G_1 \oplus G_2$ since everything is symmetric by taking the complement and exchanging LD-EMP(G) with LD-UNIV(G).

Assume first that $G_1 = K_1$. Then G_2 has at least $n_2 \geq 2$ vertices and by induction the properties $1, 2, 3$ hold for G_2. We have LD-EMP(G_1) and so LD-EMP(G). If LD-UNIV(G) holds, then by Lemma 29, we have LD-UNIV(G_2) and ¬LD-EMP(G_2), hence $sep_{LD}(G) \geq sep_{LD}(G_2) \geq \frac{n_2+1}{3} \geq \frac{n}{3}$, and we are done. Assume now that ¬LD-UNIV(G). If LD-EMP(G_2), then by Lemma 28, $sep_{LD}(G) = sep_{LD}(G_1) + sep_{LD}(G_2) + 1 \geq \frac{n_2}{3} + 1 \geq \frac{n+2}{3}$ and we are done. Otherwise, we have ¬LD-EMP(G_2) and by Lemma 29, we also have ¬LD-UNIV(G_2), hence $sep_{LD}(G) \geq sep_{LD}(G_2) \geq \frac{n_2+2}{3} = \frac{n+1}{3}$.

Assume now that none of G_1, G_2 is K_1, then by induction, the properties hold for G_1 and G_2 and we have ¬LD-UNIV(G). If both ¬LD-EMP(G_1) and ¬LD-EMP(G_2), then we also have ¬LD-EMP(G) and $sep_{LD}(G) \geq sep_{LD}(G_1) + sep_{LD}(G_2) \geq \frac{n_1+1}{3} + \frac{n_2+1}{3} \geq \frac{n+2}{3}$ and we are done. If both LD-EMP(G_1) and LD-EMP(G_2), then LD-EMP(G) and $sep_{LD}(G) = sep_{LD}(G_1) + sep_{LD}(G_2) + 1 \geq \frac{n_1}{3} + \frac{n_2}{3} + 1 = \frac{n+3}{3}$. Finally, if only one, say LD-EMP(G_1) holds, then LD-EMP(G) and $sep_{LD}(G) \geq sep_{LD}(G_1) + sep_{LD}(G_2) \geq \frac{n_1}{3} + \frac{n_2+1}{3} \geq \frac{n+1}{3}$. □

We now prove Proposition 27:

Proof of Proposition 27. We construct graphs reaching the bound by induction as follows. Assume there exist graphs G_n^1, G_n^2, G_n^3, G_n^4 on n vertices such that

- $sep_{LD}(G_n^1) = \lceil \frac{n+2}{3} \rceil$, $\neg$LD-EMP(G_n^1) and $\neg$LD-UNIV(G_n^1);
- $sep_{LD}(G_n^2) = \lceil \frac{n+1}{3} \rceil$, LD-EMP($G_n^2$) and $\neg$LD-UNIV(G_n^2);
- $sep_{LD}(G_n^3) = \lceil \frac{n+1}{3} \rceil$, $\neg$LD-EMP(G_n^3) and LD-UNIV(G_n^3);
- $sep_{LD}(G_n^4) = \lceil \frac{n}{3} \rceil$, LD-EMP($G_n^4$), LD-UNIV($G_n^4$) and G_n^4 does not have a universal vertex.

Then the graphs $G_{n+1}^1 = K_2 \oplus G_{n-1}^3$, $G_{n+1}^2 = K_1 \oplus G_n^1$, $G_{n+1}^3 = K_1 \bowtie G_n^1$, $G_{n+1}^4 = K_1 \oplus G_n^3$, satisfy the properties for $n+1$ vertices.

Starting with $G_2^2 = \overline{K_2}$, $G_2^3 = K_2$, $G_3^2 = \overline{K_3}$, $G_3^3 = K_3$, $G_4^2 = \overline{K_2} \oplus K_2$ and $G_4^3 = K_1 \bowtie (K_1 \oplus K_2)$, we obtain G_4^1, G_3^4 and G_4^4 and then graphs G_n^i for $n \geq 5$ satisfying the properties. $\square$

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