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Location-domination and matching in cubic graphs

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ABSTRACT

A dominating set of a graph G is a set D of vertices of G such that every vertex outside D is adjacent to a vertex in D. A locating-dominating set of G is a dominating set D of G with the additional property that every two distinct vertices outside D have distinct neighbors in D; that is, for distinct vertices u and v outside D, $N(u) \cap D \neq N(v) \cap D$ where $N(u)$ denotes the open neighborhood of u. A graph is twin-free if every two distinct vertices have distinct open and closed neighborhoods. The location-domination number of G, denoted $\gamma_L(G)$, is the minimum cardinality of a locating-dominating set in G. Garijo et al. (2014) posed the conjecture that for n sufficiently large, the maximum value of the location-domination number of a twin-free, connected graph on n vertices is equal to $\lfloor \frac{n}{2} \rfloor$. We propose the related (stronger) conjecture that if G is a twin-free graph of order n without isolated vertices, then $\gamma_L(G) \leq \frac{n}{2}$. We prove the conjecture for cubic graphs. We rely heavily on proof techniques from matching theory to prove our result.

1. Introduction

A *dominating set* in a graph G is a set D of vertices of G such that every vertex outside D is adjacent to a vertex in D. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set in G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [7,8]. In this paper, we focus our attention on a variation of domination, called *location-domination*, which is widely studied in the literature. A *locating-dominating set* is a dominating set D that locates all the vertices in the sense that every vertex outside D is uniquely determined by its neighborhood in D. The *location-domination number* of G, denoted $\gamma_L(G)$, is the minimum cardinality of a locating-dominating set in G. The concept of a locating-dominating set was introduced and first studied by Slater [12,13] and studied in [2,3,5,11–14] and elsewhere.

A classic result due to Ore [10] states that every graph without isolated vertices has a dominating set of cardinality at most one-half its order. As observed in [5], while there are many graphs (without isolated vertices) which have location-domination number much larger than one-half their order, the only such graphs that are known contain many *twins*, that is, pairs of vertices with the same closed or open neighborhood. Garijo, González, and Márquez [6] consider the function $\lambda_{|\mathcal{C}^*}(n)$, which is the maximum value of the location-domination number of a twin-free, connected graph on n vertices. They prove that for every $n \geq 14$, $\lambda_{|\mathcal{C}^*}(n) \geq \lfloor \frac{n}{2} \rfloor$, and they find different conditions for a twin-free graph G to satisfy $\gamma_L(G) \leq \lfloor \frac{n}{2} \rfloor$. Motivated by these results, they state the following conjecture.

Conjecture 1 ([6]). *There exists a positive integer n_1 such that, for every $n \geq n_1$, $\lambda_{|\mathcal{C}^*}(n) = \lfloor \frac{n}{2} \rfloor$.*

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We pose the related conjecture that in the absence of twins, the classic bound of one-half the order for the domination number also holds for the location-domination number.

Conjecture 2. *Every twin-free graph G of order n without isolated vertices satisfies* $\gamma_L(G) \le \frac{n}{2}$.

We remark that Conjecture 2 implies Conjecture 1. Indeed, if Conjecture 2 is true, then $\lambda_{|\mathcal{C}^*}(n) \le \lfloor \frac{n}{2} \rfloor$ for all $n \ge 2$, which implies, by the results of Garijo et al. [6], that $\lambda_{|\mathcal{C}^*}(n) = \lfloor \frac{n}{2} \rfloor$ for every $n \ge 14$. Moreover, Conjecture 2 is a stronger conjecture than Conjecture 1 in the sense that Conjecture 2 applies to twin-free graphs of arbitrary order with no isolated vertex, while Conjecture 1 is claimed to hold only for (connected) twin-free graphs of sufficiently large order.[1]

Strict inequality may hold in Conjecture 2. Consider, for example, the twin-free, bipartite graph G formed by taking as one partite set a set S of $k \ge 2$ elements, and as the other partite set all the distinct non-empty subsets of S, and joining each element of S to those subsets it is a member of. Then, G has order $n = k + 2^k - 1$ and $\gamma_L(G) = |S| = k = \lfloor \log_2 n \rfloor$. This is a classic construction in the area of location-domination, see for example [13].

Garijo et al. [6] prove Conjecture 2 for graphs without 4-cycles (which include trees) and for the class of graphs with independence number at least one-half the order (which includes bipartite graphs). Further, they prove Conjecture 2 for twin-free graphs satisfying certain conditions on the upper domination number and the chromatic number. In [5], the authors provide several constructions for twin-free graphs with location-domination number one-half their order. The variety of these constructions shows that these graphs have a rich structure, which is an indication that Conjecture 2 might be difficult to prove. Further support is given to this conjecture in [5] where it is proved for split graphs and co-bipartite graphs, and in [4] where it is proved for line graphs. The following theorem summarizes the known results about Conjecture 2.

Theorem (*[5,6,9]*). *Conjecture 2 is true if the twin-free graph G of order n (without isolated vertices) satisfies any of the following conditions.*

(a) [6] *G has no 4-cycles.*

(b) [6] *G has independence number at least* $\frac{n}{2}$.

(c) [6] *G has clique number at least* $\lceil \frac{n}{2} \rceil + 1$.

(g) [6] *G has upper domination number at least* $\frac{n}{2}$ *or* $\overline{G}$ *has upper domination number at least* $\frac{n}{2} + 1$.

(h) [6] *G has chromatic number at least* $\frac{3n}{4}$ *or* $\overline{G}$ *has chromatic number at least* $\frac{3n}{4} + 1$.

(d) [5] *G is a split graph or a co-bipartite graph.*

(e) [4] *G is a line graph.*

(f) [9] *G is a claw-free, cubic graph.*

In this paper, we continue to advance the study of Conjecture 2 by proving it for the class of cubic graphs, as stated in our main theorem:

Theorem 3. *If G is a twin-free, cubic graph of order n, then* $\gamma_L(G) \le \frac{n}{2}$.

We start by giving some definitions and notations in Section 2, and we prove Theorem 3 in Section 3. The essence of our proof of Theorem 3 is to apply the Tutte–Berge Formula and use matching theory in order to obtain certain desired structures of a cubic graph that will enable us to construct locating-dominating sets of size at most one-half the order of the graph.

2. Definitions and notation

For notation and graph theory terminology, we in general follow [7]. Specifically, let G be a graph with vertex set $V(G)$, edge set $E(G)$ and with no isolated vertex. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of v is $d_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write V, E, $N(v)$, $N[v]$ and $d(v)$ rather than $V(G)$, $E(G)$, $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively. Two distinct vertices u and v of a graph G are *open twins* if $N(u) = N(v)$ and *closed twins* if $N[u] = N[v]$. Further, u and v are *twins* in G if they are open twins or closed twins in G. A graph is *twin-free* if it has no twins. We use the standard notation $[k] = \{1, 2, \dots, k\}$.

Given a set F of edges, we will denote by $G - F$ the subgraph obtained from G by deleting all edges of F. For a set S of vertices, $G - S$ is the graph obtained from G by removing all vertices of S and removing all edges incident to vertices of S. The subgraph induced by S is denoted by $G[S]$. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. An *odd component* of G is a component of G of odd order. The number of odd components of G is denoted by $\text{oc}(G)$.

[1] In [5], we attributed Conjecture 2 to the authors of [6] who posed Conjecture 1. However, as correctly pointed out by the reviewers of the current paper, the statements of Conjectures 1 and 2 are different. Hence, although Conjecture 2 is motivated by Conjecture 1, we pose Conjecture 2 as an independent conjecture which is a strengthening of Conjecture 1.

A set D is a dominating set of G if $N[v] \cap D \neq \emptyset$ for every vertex v in G, or, equivalently, $N[S] = V(G)$. Two distinct vertices u and v in $V(G) \setminus D$ are *located* by D if they have distinct neighbors in D; that is, $N(u) \cap D \neq N(v) \cap D$. If a vertex $u \in V(G) \setminus D$ is located from every other vertex in $V(G) \setminus D$, we simply say that u is *located* by D. For $k \geq 1$ if X is a set of vertices in G and $x \in V(G) \setminus X$, then the vertex x is said to be k-dominated by X if x has exactly k neighbors inside X; that is, $|N(x) \cap X| = k$.

A set S is a *locating set* of G if every two distinct vertices outside S are located by S. In particular, if S is both a dominating set and a locating set, then S is a locating-dominating set. Further, if S is both a total dominating set and a locating set, then S is a *locating-total dominating set* (where S is a *total dominating set* of G if every vertex of G is adjacent to some vertex in S). We remark that the only difference between a locating set and a locating-dominating set in G is that a locating set might have a unique non-dominated vertex.

An *independent set* in G is a set of vertices no two of which are adjacent. Two distinct edges in a graph G are *independent* if they are not adjacent in G (i.e., the two edges are not incident with a common vertex). A set of pairwise independent edges of G is called a *matching* in G. A matching of maximum cardinality in G is called a *maximum matching* in G. The number of edges in a maximum matching of a graph G is called the *matching number* of G, denoted by $\alpha'(G)$. Let M be a specified matching in a graph G. A vertex v of G is an *M-matched vertex* if v is incident with an edge of M; otherwise, v is an *M-unmatched vertex*. If the matching M is clear from context, we simply call a M-matched vertex a *matched vertex* and a M-unmatched vertex an *unmatched vertex*.

3. Proof of Theorem 3

In this section, we present a proof of Theorem 3. Our proof relies heavily on matching theory in graphs. We begin with some useful definitions and lemmas related to matchings.

3.1. Useful definitions and lemmas

We shall need the following theorem of Berge [1] about the matching number of a graph, which is sometimes referred to as the Tutte–Berge formulation for the matching number. Recall that $\mathrm{oc}(G)$ denotes the number of odd components in a graph G.

Theorem 4 (*Tutte–Berge Formula*)**.** *For every graph G,*

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2} \left(|V(G)| + |X| - \mathrm{oc}(G - X)\right).$$

We shall also need the following structural result about maximum matchings in graphs which is a consequence of the proof of the Tutte–Berge Formula.

Theorem 5 ([1])**.** *Let $G = (V, E)$ be a graph and let X be a proper subset of vertices of G such that $(|V| + |X| - \mathrm{oc}(G - X))/2$ is minimum. If M is a maximum matching in G, then $|M| = (|V| + |X| - \mathrm{oc}(G - X))/2$ and there are exactly $\mathrm{oc}(G - X) - |X|$ vertices that are M-unmatched. Furthermore, if M_X is the subset of edges of M that belong to $G - X$, then every vertex in $G - X$ is M_X-matched, except for exactly one vertex from each odd component of $G - X$. If U denotes this set of $\mathrm{oc}(G - X)$ vertices that are M_X-unmatched, one from each odd component of $G - X$, then X is M-matched to a subset of vertices in U.*

The structure described in Theorem 5 is illustrated in Fig. 1.

Definition of the set $\mathcal{D}_G(M)$. Let G be a graph and let M be a maximum matching of G. We define $\mathcal{D}_G(M)$ to be the collection of all sets D of vertices such that the following holds:

- For every edge $uv \in M$, if exactly one of u and v has a neighbor that is M-unmatched, then the vertex in $\{u, v\}$ with an M-unmatched neighbor belongs to D.
- For every edge $uv \in M$, if neither u nor v has an M-unmatched neighbor or if both u and v have a (common) M-unmatched neighbor, then exactly one of u and v belongs to D.

If the graph G is clear from the context, we simply write $\mathcal{D}(M)$ rather than $\mathcal{D}_G(M)$.

Definition of a D-bad pair. Given a set $D \subseteq V(G)$, we define a *D-bad pair* of vertices as two vertices in $V(G) \setminus D$ that are not located by D. If the set D is clear from the context, we simply write that a pair of vertices is a *bad pair* rather than a D-bad pair.

In our proof, we will use the following lemmas.

Lemma 6. *Let G be a cubic graph, let M be a maximum matching of G, and let $D \in \mathcal{D}_G(M)$. Then, D is a dominating set of G, and each M-unmatched vertex is dominated by at least two vertices of D.*

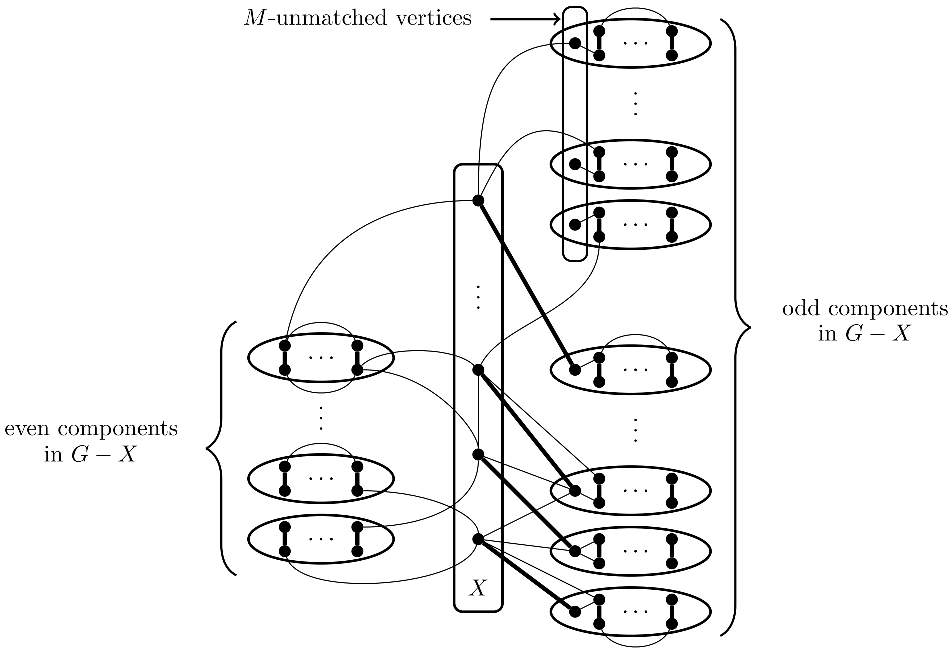


Fig. 1. Example of the structure of a graph with maximum matching M (thickened edges) with respect to a given set X.

Proof. It follows readily from the two properties of sets $D \in \mathcal{D}_G(M)$, that every M-matched vertex is dominated by D. If x is an M-unmatched vertex, then since G is cubic, the vertex x is adjacent to two M-matched vertices that are incident with distinct edges, e_1 and e_2 say, of M. Hence, by the construction of D, the set D contains a neighbor of x incident with e_1 and a neighbor of x incident with e_2. Thus, x is dominated by at least two vertices of D. □

Lemma 7. *Let G be a twin-free, cubic graph, let M be a maximum matching of G, and let $D \in \mathcal{D}_G(M)$. Then, the vertices of each D-bad pair are 2-dominated.*

Proof. Let $\{u, v\}$ be a D-bad pair. If u and v were 3-dominated by D, then they would be open twins, a contradiction. Hence, u and v are dominated by at most two vertices of D. By Lemma 6, only M-matched vertices can be 1-dominated by D, but if u and v are M-matched vertices, they would each be dominated by the vertex of D that they are matched to under M and would therefore not form a D-bad pair, a contradiction. Hence, u and v are 2-dominated by D. □

Lemma 8. *Let G be a twin-free, cubic graph. Among all maximum matchings M of G and all sets $D \in \mathcal{D}_G(M)$, let the matching M_0 and the set $D_0 \in \mathcal{D}_G(M_0)$ be chosen so that the number of D_0-bad pairs is minimum. Then, the vertices of each D_0-bad pair are M_0-matched vertices.*

Proof. Let X be a proper subset of vertices of G such that $(|V|+|X|-\mathrm{oc}(G-X))/2$ is minimum. The structure of the graph G with respect to the matching M_0 and the set X is described in Theorem 5. Let $\{u, v\}$ be an arbitrary D_0-bad pair. Suppose to the contrary that they are not both M_0-matched vertices. By Lemma 7, both u and v are 2-dominated by D_0. By the definition of a set in $\mathcal{D}_G(M_0)$, if both u and v are M_0-unmatched and 2-dominated, they would be open twins, a contradiction. Therefore, exactly one of u and v is M_0-unmatched. Renaming u and v if necessary, we may assume that u is M_0-unmatched. Thus, by Theorem 5, the vertex u belongs to an odd component, C_u say, of $G - X$. Let x and y be the two common neighbors of u and v in D_0. Let x' and y' be the vertices M_0-matched to x and y, respectively.

Suppose that both x and y belong to C_u. If $v \notin V(C_u)$, then $v \in X$. But then the vertex that is M_0-matched to v belongs to D_0, implying that v would be 3-dominated by D_0, contradicting Lemma 7. Hence, $v \in V(C_u)$. If v is matched to neither x nor y by M_0, then, once again, v would be 3-dominated by D_0, a contradiction. Hence, v is M_0-matched to either x or y. Renaming x and y if necessary, we may assume that $xv \in M_0$, and so $v = x'$. If u and v are adjacent, then u and v would be closed twins, a contradiction. Hence, u and v are not adjacent. The third neighbor of u, different from x and y, is therefore the vertex y' that is M_0-matched to y (otherwise, by the definition of $\mathcal{D}_G(M)$, u would be 3-dominated, a contradiction). We now consider the set $D' = (D_0 \setminus \{y\}) \cup \{y'\}$.

We note that y and y' have a common M_0-unmatched neighbor, namely u, implying that $D' \in \mathcal{D}(M_0)$. Suppose there is a vertex z different from u that is adjacent to both x and y'. Then, z is either in X or in C_u. In both cases, z is M_0-matched. If $z = v$, then u and z are open twins, a contradiction. Since $N(y) = \{u, v, y'\}$ while z is adjacent to x, we note that $z \neq y$. Therefore, $z \notin \{v, y\}$ and the M_0-matched neighbor of z is in D', implying that z is 3-dominated by D'. Hence, the vertex u is the only vertex dominated only by x and y' in D' and it is therefore located by D'. Moreover, both v and y are 1-dominated

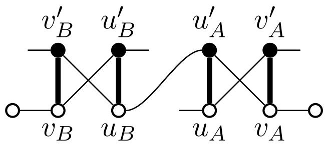


Fig. 2. Two bad (D, M)-matched 4-cycles A and B, where A is dependent on B via u'_A. Edges of M are thickened. Black vertices belong to D, and white vertices do not.

by D', and are therefore located by D' by Lemma 7. Finally, no other vertex has been affected by the removal of y from D_0. Hence, the number of D'-bad pairs is strictly less than the number of D_0-bad pairs, contradicting our choice of D_0. Therefore, at most one of x and y belong to C_u.

Suppose that exactly one of x and y belongs to C_u. Renaming x and y if necessary, we may assume that $x \in V(C_u)$. Then, $y \in X$. If $v \in X$ or if $v \in V(C_u) \setminus \{x'\}$, then v would be 3-dominated by D_0, a contradiction. Hence, $v = x'$, and so v is M_0-matched to x. Since u is 2-dominated by D_0, the vertex u is adjacent to either v or y'. Since y' belongs to a component of $G - X$ different from C_u, the vertex u is adjacent to v. But then u and v are closed twins, a contradiction.

Therefore, both x and y belong to X. This implies that u, x' and y' belong to three different components of $G - X$. In particular, u is adjacent to neither x' nor y', implying that the third neighbor of u different from x and y is an M_0-matched vertex and therefore belongs to the set D_0 by the construction of sets in $\mathcal{D}_G(M_0)$. Thus, u is then 3-dominated by D_0, a contradiction. This completes the proof of the lemma. □

Note that in any twin-free, cubic graph G, every 4-cycle is an induced 4-cycle. The following structure will play an important role in our proof.

Definition of a bad (D, M)-matched 4-cycle. Let $C: u_C u'_C v_C v'_C u_C$ be a 4-cycle in a (twin-free cubic) graph G, M a matching of G, and D a subset of vertices of G. We say that C is a *bad (D, M)-matched 4-cycle* if $u_C u'_C \in M$, $v_C v'_C \in M$, $D \cap V(C) = \{u'_C, v'_C\}$ and v_C is adjacent to exactly two vertices of D (and so, $N(v_C) \cap D = \{u'_C, v'_C\}$).

Given two bad (D, M)-matched 4-cycles, A and B, we say that A is *dependent* on B via the vertex u'_A or v'_A if u_B is adjacent to u'_A or to v'_A, respectively. An illustration is given in Fig. 2. We note that if A is dependent on B, then u_B is 3-dominated by D.

Given a set S of vertex-disjoint bad (D, M)-matched 4-cycles of a graph G, let $\overrightarrow{G}(S)$ be the digraph with vertex set S and where (A, B) is an arc in $\overrightarrow{G}(S)$ if A is dependent on B. We remark that since G is cubic, every vertex in $\overrightarrow{G}(S)$ has out-degree at most 2. Further by definition of a bad (D, M)-matched 4-cycle, every vertex in $\overrightarrow{G}(S)$ has in-degree at most 1.

Given a rooted tree T with root r, by an *orientation of* T we mean orienting every arc of T from a parent to its child.

3.2. *Proof of the main result*

We are now in a position to prove our main result, namely Theorem 3. Recall its statement.

Theorem 3. *If G is a twin-free, cubic graph of order n, then $\gamma_L(G) \le \frac{n}{2}$.*

Proof of Theorem 3. Among all maximum matchings M of G and all sets $D \in \mathcal{D}_G(M)$, we choose the matching M_0 and the set $D_0 \in \mathcal{D}_G(M_0)$ so that the number of D_0-bad pairs is minimum. Let X be a proper subset of vertices of G such that $(|V|+|X|-\mathrm{oc}(G-X))/2$ is minimum. The structure of the graph G with respect to the matching M_0 and the set X is described in Theorem 5.

We now describe the structure of D_0-bad pairs:

Claim A. *Every D_0-bad pair $\{u, v\}$ belongs to a common bad (D_0, M_0)-matched 4-cycle, say R. Further, there exists a set $S_{u,v}$ of vertex-disjoint bad (D_0, M_0)-matched 4-cycles containing R such that the following holds:*

(a) *For every 4-cycle $C \in S_{u,v}$ and every vertex $x \in \{u'_C, v'_C\}$, either x is adjacent to an M_0-unmatched vertex in G, or C is dependent on some other $C' \in S_{u,v}$ via the vertex x.*

(b) *$\overrightarrow{G}(S_{u,v})$ is an oriented tree rooted at R.*

(c) *For every 4-cycle $C \in S_{u,v}$, if both u'_C and v'_C have an M_0-unmatched neighbor, then these neighbors are distinct and $\{u'_C, v'_C\} \subseteq X$.*

Proof of Claim A. Let $\{u, v\}$ be a D_0-bad pair. Thus, u and v are vertices outside D_0 that are not located by D_0. By Lemma 7, both u and v are 2-dominated by D_0, and by Lemma 8, both u and v are M_0-matched. Let u' and v' be the M_0-matched neighbors of u and v, respectively. Since u and v are 2-dominated by D_0, we note that u' and v' are the two common neighbors of u and v in D_0. Thus, $C_{uv}: uu'vv'u$ is a bad (D_0, M_0)-matched 4-cycle in G. As observed earlier, every 4-cycle in G is an induced 4-cycle. Hence, let x, y, u'' and v'' be the neighbors of u, v, u' and v', respectively, that do not belong to this 4-cycle C_{uv}. Let $D_u = (D_0 \setminus \{u'\}) \cup \{u\}$ and let $D_v = (D_0 \setminus \{v'\}) \cup \{v\}$. We proceed further with the following series of subclaims.

Claim A.1. *The following holds.*

(a) *If $D_u \notin \mathcal{D}_G(M_0)$, then u'' is an M_0-unmatched vertex that is not adjacent to u.*

(b) *If $D_u \in \mathcal{D}_G(M_0)$, then the only D_u-bad pair that is not a D_0-bad pair is $\{u'', z\}$ for some vertex z. Moreover, u'' and z are part of a bad (D_0, M_0)-matched 4-cycle C of G, and C_{uv} is dependent on C via the vertex u'.*

Proof of Claim A.1. By definition of the sets in the family $\mathcal{D}_G(M_0)$, if $D_u \notin \mathcal{D}_G(M_0)$, then u'' is an M_0-unmatched vertex and u'' is not adjacent to u, proving Statement (a) of Claim A.1.

To prove Statement (b), suppose that $D_u \in \mathcal{D}_G(M_0)$. By our choice of M_0 and D_0, there are at least as many D_u-bad pairs as D_0-bad pairs. The only vertices that could potentially be negatively affected (in the sense that they are located by D_0 but not by D_u) by removing u' from D_0 and replacing it with the vertex u are u', u'' and v. By Lemma 7, two vertices forming a D_u-bad pair are 2-dominated by D_u. The vertex v is 1-dominated by D_u, and hence it is located by D_u.

Suppose that u' is not located by D_u from some other vertex outside D_u. Then, this vertex must be x, the neighbor of u not on C_{uv}. Considering the D_u-bad pair $\{u', x\}$, and noting that u' and x are 2-dominated by D_u, we deduce that $u'' \in D_u$. If x is M_0-unmatched, then by the definition of $\mathcal{D}_G(M_0)$, we would have $u \in D_0$ and $u' \notin D_0$, a contradiction. Hence, x is M_0-matched and its matched neighbor is in D_u. Since x is 2-dominated, we have $xu'' \in M_0$. We now consider the maximum matching $M' = (M_0 \setminus \{uu', vv'\}) \cup \{uv', vu'\}$, and we let $D' = (D_0 \setminus \{u'\}) \cup \{v\}$.

We note that $u''x \in M'$. Since neither u nor u' has an M'-unmatched neighbor, we note that $D' \in \mathcal{D}(M')$. The only vertices that could potentially be negatively affected (in the sense that they are located by D_0 but not by D') by these changes are the vertices dominated by u' in D_0 and that do not belong to D'. The only such vertices are u and u'. By Lemma 7, if two vertices form a D'-bad pair, then they are 2-dominated by D'. The vertex u is 1-dominated by D', and hence it is located by D'. Thus, the vertex u' is not located by D' from some other vertex outside D'. Such a vertex must be adjacent to both v and u'', and is therefore the neighbor of v outside C_{uv}, namely the vertex y. If $x = y$, then u and v would be open twins, a contradiction. Hence, $x \neq y$. If y is M'-matched, since $u''x \in M'$ and $u'v \in M'$, the vertex y is 3-dominated by D', a contradiction. Hence, y is M'-unmatched (and M-unmatched). Then, since $D_0 \in \mathcal{D}_G(M_0)$, the vertices y and v' are adjacent. Hence, y is 3-dominated by D', a contradiction. Thus, u' is located by D_u.

Therefore, among the vertices dominated by u' in D_0, the vertex u'' is the only vertex that was located by D_0 but that is not located by D_u from some other vertex, z say, outside D_u. Thus, $\{u'', z\}$ is the only pair of vertices located by D_0 but not by D_u. Hence, the number of D_u-bad pairs is the same as the number of D_0-bad pairs (since $\{u, v\}$ is not a D_u-bad pair) and we can apply Lemma 8 to D_u to deduce that u'' and z are M_0-matched vertices. By Lemma 7, both u'' and z are 2-dominated by D_u. Let w and t be the M_0-matched neighbors of u'' and z, respectively. Since u'' and z are 2-dominated by D_u and $D_u \in \mathcal{D}_G(M_0)$, we note that w and t are the two common neighbors of u'' and z in D_u. Thus, the 4-cycle $C : u''wztu''$ is a bad (D_0, M_0)-matched 4-cycle in G with $u'' = u_C$, $w = u'_C$, $z = v_C$ and $t = v'_C$, and C_{uv} is dependent on C via the vertex u'. This establishes Statement (b) of Claim A.1. ◇

Interchanging the roles of u and v in the proof of Claim A.1, we have the following analogous result for the vertex v.

Claim A.2. *The following holds.*

(a) *If $D_v \notin \mathcal{D}_G(M_0)$, then v'' is an M_0-unmatched vertex that is not adjacent to v.*

(b) *If $D_v \in \mathcal{D}_G(M_0)$, then the only D_v-bad pair that is not a D_0-bad pair is $\{v'', z\}$ for some vertex z. Moreover, v'' and z are part of a bad (D_0, M_0)-matched 4-cycle C of G, and C_{uv} is dependent on C via the vertex v'.*

Let R denote the bad (D_0, M_0)-matched 4-cycle $C_{uv} : uu'vv'u$, where $u = u_R$, $u' = u'_R$, $v = v_R$ and $v' = v'_R$. We now show the existence of a set $S_{u,v}$ of vertex-disjoint bad (D_0, M_0)-matched 4-cycles containing R such that conditions (a) and (b) in the statement of Claim A hold. If both u' and v' have an M_0-unmatched neighbor, then we let $S_{u,v} = \{R\}$ and we are done.

Otherwise, renaming vertices if necessary, we may assume, by Claims A.1 and A.2, that $D_u \in \mathcal{D}_G(M_0)$ and that R is dependent on a bad (D_0, M_0)-matched 4-cycle C via vertex u'_R. Now, since by Claim A.1(b) the number of D_u-bad pairs is the same as the number of D_0-bad pairs (hence D_u also minimizes the number of bad pairs), we can apply Claim A.1 and Claim A.2 to C, D_u and to the D_u-bad pair $\{u_C, v_C\}$. This shows that each of u'_C and v'_C either have an M_0-unmatched neighbor, or C is dependent on some other bad (D_0, M_0)-matched 4-cycle via this vertex. Repeating this process as long as possible yields a set $S_{u,v}$ of bad (D_0, M_0)-matched 4-cycles, where for each bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ different from R there is some other bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ that depends on it, and satisfies the properties in Claims A.1 and A.2. This establishes Statement (a) of Claim A. Moreover we have the following.

Claim A.3. *Any two distinct 4-cycles in $S_{u,v}$ are vertex-disjoint.*

Proof of Claim A.3. Let $A : u_Au'_Av_Av'_A$ and $B : u_Bu'_Bv_Bv'_B$ be two distinct 4-cycles of $S_{u,v}$. If they have a common vertex, since their vertices are pairwise M_0-matched, they must have two vertices in common, and these vertices must be M_0-matched to each other. But then, the vertex that belongs to both A and B but does not belong to D_0 must be 3-dominated by D_0, a contradiction. ◇

Now, consider the digraph $\overrightarrow{G}(S_{u,v})$, which by Claim A.3 is well-defined. The following properties hold in the digraph $\overrightarrow{G}(S_{u,v})$. Recall that the distance from a vertex x to a vertex y in a directed graph D is the minimum length among all directed paths from x to y in D.

Claim A.4. *The following holds.*

(a) *R has in-degree* 0 *in* $\overrightarrow{G}(S_{u,v})$.

(b) *Every vertex in* $\overrightarrow{G}(S_{u,v})$ *different from R has in-degree exactly* 1.

(c) $\overrightarrow{G}(S_{u,v})$ *has no directed cycle.*

Proof of Claim A.4. To see that Statement (a) holds, observe that if some bad (D_0, M_0)-matched 4-cycle C was dependent on R say, on vertex u_R, then u_R would be 3-dominated by D_0, a contradiction.

For Statement (b), we show firstly that $\overrightarrow{G}(S_{u,v})$ has maximum in-degree 1. Suppose to the contrary that for some bad (D_0, M_0)-matched 4-cycle A in $S_{u,v}$, there are two other bad (D_0, M_0)-matched 4-cycles B and C of $S_{u,v}$ that are both dependent on A. Since B is dependent on A, the vertex u_A is adjacent to u'_B or v'_B. Since C is dependent on A, the vertex u_A is adjacent to u'_C or v'_C. Thus, the vertex u_A has degree at least 4, a contradiction. We show secondly that R is the only vertex in $\overrightarrow{G}(S_{u,v})$ with in-degree 0. As observed in the paragraph immediately preceding Claim A.3, for each bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ different from R there is some other bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ that depends on it. Therefore, every bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ different from R has in-degree at least 1 in $\overrightarrow{G}(S_{u,v})$. Therefore, by our earlier observations, every bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ different from R has in-degree exactly 1 in $\overrightarrow{G}(S_{u,v})$.

For Statement (c), suppose to the contrary that $\overrightarrow{G}(S_{u,v})$ contains a directed cycle $C: C_1C_2 \dots C_kC_1$ for some $k \geq 2$. By Statement (a), we know that R has in-degree 0 and therefore cannot belong to this cycle. However, there is a (directed) path from R to every other vertex in $\overrightarrow{G}(S_{u,v})$. Among all vertices in the directed cycle C, let C_i be chosen so that the distance from R to C_i in $\overrightarrow{G}(S_{u,v})$ is minimum where $i \in [k]$. Let P be a shortest (directed) path from R to C_i in $\overrightarrow{G}(S_{u,v})$ and let B be the vertex that immediately precedes C_i on the path P (possibly, $B = R$). Since the distance from R to B in $\overrightarrow{G}(S_{u,v})$ is less than the distance from R to C_i in $\overrightarrow{G}(S_{u,v})$, the vertex B does not belong to the directed cycle C. Therefore, C_i has in-degree at least 2 in $\overrightarrow{G}(S_{u,v})$, contradicting Statement (b). This completes the proof of the claim. $\diamond$

By Claim A.4(c), if there is a cycle in $\overrightarrow{G}(S_{u,v})$, it cannot be an oriented cycle. But then some vertex in that cycle must have in-degree at least 2, contradicting Claim A.4(b). Hence, Claim A.4 implies that $\overrightarrow{G}(S_{u,v})$ is an oriented tree rooted at R, and we have proved Statement (b) of Claim A.

It remains to prove Statement (c) of Claim A. Let $C \in S_{u,v}$ be a bad (D_0, M_0)-matched 4-cycle where both u'_C and v'_C have an M_0-unmatched neighbor, u'' and v'' say, respectively. These neighbors are clearly distinct, since otherwise u'_C and v'_C are open twins. By Theorem 5, the two M_0-unmatched vertices u'' and v'' belong to distinct odd components of $G - X$. Suppose to the contrary that $u_C \in X$. Then, by Theorem 5, $u'_C \notin X$ and u'_C belongs to an odd component of $G - X$ that contains no M_0-unmatched vertex. However, u'_C is adjacent to the M_0-unmatched vertex u'' which implies that u'_C belongs to the same odd component of $G - X$ as the M_0-unmatched vertex u'', a contradiction. Hence, $u_C \notin X$. Analogously, $v_C \notin X$.

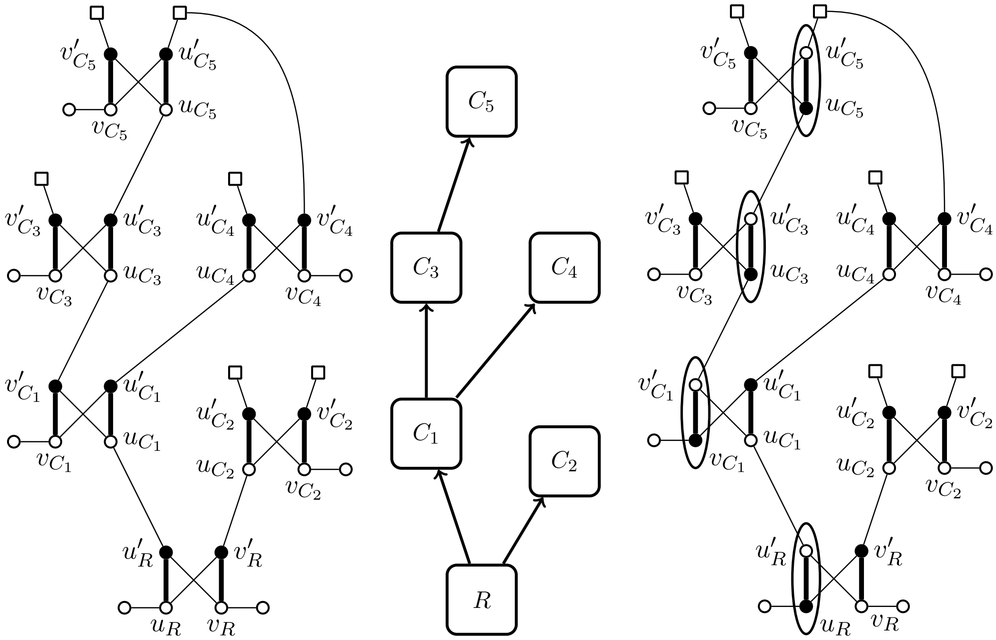
If neither u'_C nor v'_C belongs to X, then u'' and v'' belong to the same components of $G - X$, a contradiction. Hence, renaming u'_C and v'_C, if necessary, we may assume that $v'_C \in X$. Thus, v_C belongs to an odd component of $G - X$ that contains no M_0-unmatched vertex. If $u'_C \notin X$, then v_C would be in the same odd component of $G - X$ as the M_0-unmatched vertex u'', a contradiction. Hence, $u'_C \in X$. This establishes Statement (c) of Claim A and completes the proof of Claim A. $\diamond$

An example of a subgraph of G corresponding to a set $S_{u,v}$ that contains six bad (M_0, D_0)-matched 4-cycles is illustrated in Fig. 3(a) (where the edges of M_0 are thickened, black vertices belong to D_0 and white vertices do not, and square white vertices are M_0-unmatched vertices).

We now return to the proof of Theorem 3. Our strategy is to modify the set D_0 in such a way that the resulting set becomes a locating-dominating set of G of cardinality at most one-half the order of G. Consider the set of D_0-bad pairs. By Claim A, each such D_0-bad pair $\{u, v\}$ belongs to a bad (D_0, M_0)-matched 4-cycle R and there is a set $S_{u,v}$ of vertex-disjoint bad (D_0, M_0)-matched 4-cycles such that $\overrightarrow{G}(S_{u,v})$ is an oriented tree rooted at R. We note that, given two D_0-bad pairs $\{u, v\}$ and $\{x, y\}$, the trees $\overrightarrow{G}(S_{u,v})$ and $\overrightarrow{G}(S_{x,y})$ are vertex-disjoint, and furthermore no bad (D_0, M_0)-matched 4-cycles of $\overrightarrow{G}(S_{u,v})$ and $\overrightarrow{G}(S_{x,y})$ share any vertex. Indeed, by similar arguments as in Claims A.3 and A.4, we would otherwise contradict the definition of a bad (D_0, M_0)-matched 4-cycle.

Now, given a D_0-bad pair $\{u, v\}$, consider any leaf C of $\overrightarrow{G}(S_{u,v})$. By Claim A(a), the vertices u'_C and v'_C both have a distinct M_0-unmatched neighbor, say u''_C and v''_C, respectively. Further, by Claim A(c), both u'_C and v'_C belong to X.

For each D_0-bad pair $\{u, v\}$, we select an arbitrary leaf C of $\overrightarrow{G}(S_{u,v})$ and associate the pair of vertices $u'' = u''_C$ and $v'' = v''_C$ of M_0-unmatched neighbors of u'_C and v'_C, respectively, with the pair $\{u, v\}$, and we write $f(u, v) = \{u'', v''\}$. Let V^* be the set of all M_0-unmatched vertices associated with some D_0-bad pair. We define the (multi)graph G^* on the vertex set V^* by adding an edge joining u'' and v'' for each D_0-bad pair $\{u, v\}$ such that $f(u, v) = \{u'', v''\}$. As remarked earlier, the vertices u'' and v'' are distinct, implying that G^* has no loops (although it may have multiple edges), no isolated vertices, and is subcubic (that is, has maximum degree at most 3). Our aim is to add at most $|V^*|/2$ vertices to D_0 and to locally modify D_0 around



(a) The structure around the D_0-bad pair.

(b) The oriented tree $\overrightarrow{G}(S_{u_R,v_R})$.

(c) The modification of D_0 around $\{u_R, v_R\}$ when associating it with the leaf C_5 of $\overrightarrow{G}(S_{u_R,v_R})$. The vertices of each circled pair get swapped in D_0.

Fig. 3. Example of a D_0-bad pair $\{u_R, v_R\}$ with the set of bad (D_0, M_0)-matched 4-cycles $S_{u_R,v_R} = \{R, C_1, \ldots, C_5\}$. The edges of M_0 are thickened; squared vertices are M_0-unmatched; black vertices belong to D_0.

the D_0-bad pairs in order to obtain a locating-dominating set, D', of cardinality

$$|D'| \le \alpha'(G) + \frac{|V^*|}{2} \le \alpha'(G) + \frac{n - 2\alpha'(G)}{2} = \frac{n}{2}.$$

We now describe the construction of such a set D'. Let D^* be a minimum dominating set of G^*. Since G^* has no isolated vertex, $|D^*| \le |V^*|/2$. Since G^* has maximum degree at most 3 and since every vertex outside D^* is adjacent to at least one vertex of D^* in G^*, we note that $G^* - D^*$ has maximum degree at most 2. We now build a locating-dominating set from D_0 by adding D^* to D_0 and by propagating modifications of D_0 along the oriented trees associated with all D_0-bad pairs. More precisely, we perform our propagation as follows.

Step 1: We first consider all D_0-bad pairs associated with a pair of vertices of G^* at least one vertex of which belongs to the set D^*. Let $\{u, v\}$ be such a D_0-bad pair, and let u'' and v'' be the vertices of V^* such that $f(u, v) = \{u'', v''\}$. Adopting our earlier notation, let $u'' = u''_C$ and $v'' = v''_C$, where C is the chosen leaf in the tree $\overrightarrow{G}(S_{u,v})$. Let R be the bad (D_0, M_0)-matched 4-cycle in $S_{u,v}$ containing u and v. Renaming u'' and v'', if necessary, we may assume that u'' belongs to D^*. We now consider the unique (directed) path P of $\overrightarrow{G}(S_{u,v})$ joining R to C and we modify D_0 along P as follows. First, replace u'_C with u_C in D_0. If B is the parent of C in $\overrightarrow{G}(S_{u,v})$ (and so, B is the vertex on the (R, C)-path P that immediately precedes C) and B is dependent on C via x'_B, where $x_B \in \{u_B, v_B\}$, we replace x'_B with x_B in D_0. We continue this process until we perform the modification in the root R. This exchange argument in the oriented tree $\overrightarrow{G}(S_{u,v})$ associated with the subgraph of G corresponding to the set $S_{u,v}$ illustrated in Fig. 3(a) is shown in Fig. 3(c). This process is done for all D_0-bad pairs associated with a pair of vertices of G^* with at least one member in D^*. Let D' be the resulting modified set D_0.

Claim B. *The set of $(D' \cup D^*)$-bad pairs is a proper subset of the set of D_0-bad pairs.*

Proof of Claim B. Let $\{u, v\}$ be an original D_0-bad pair associated with a pair of vertices of G^* at least one of which belongs to the set D^*. Since at least one of u and v now belongs to D', the pair $\{u, v\}$ is not a $(D' \cup D^*)$-bad pair. It suffices to check the pairs of vertices that could possibly have been affected by the exchange arguments; that is, all vertices previously dominated by a vertex that has been removed from D_0 to construct D' (this includes all vertices removed from D_0 to construct D'). A vertex affected by the modification belongs to a bad (D_0, M_0)-matched 4-cycle A in the selected path of $\overrightarrow{G}(S_{u,v})$ for some D_0-bad pair $\{u, v\}$ that is associated with a pair of vertices of G^* at least one of which belongs to the set D^*. Let the vertex set of A be $\{x_A, y_A, x'_A, y'_A\}$ with $\{x_A, y_A\} = \{u_A, v_A\}$, where x'_A has been replaced with x_A in D'.

It is sufficient to check that the vertices x'_A, y_A and the neighbor, z say, of x'_A not in A are located by $D' \cup D^*$ or belong to $D' \cup D^*$. We observe that even though D' might not belong to $\mathcal{D}(M)$, the set D' contains exactly one vertex from each edge in M_0. We note that every vertex that was removed from D_0 during the exchange arguments when constructing D' is either adjacent to a vertex of D^* or is adjacent to no M_0-unmatched vertex. Hence, the vertices of D_0 that are adjacent to an M_0-unmatched vertex that does not belong to D^* are not removed from D_0 during the exchange arguments, implying by Lemma 6 that every M_0-unmatched vertex is adjacent to at least two vertices in $D' \cup D^*$ or belongs to D^*. It follows that every vertex that is 1-dominated by $D' \cup D^*$ is located by this set. In particular, irrespective of whether $y = u$ or $y = v$, the vertex y_A is 1-dominated by $D' \cup D^*$ and is thus located by $D' \cup D^*$.

Suppose that z is an M_0-unmatched vertex. Then, by construction, $z \in D^*$. In this case, x'_A is dominated by z and x_A but by no other vertex in $D' \cup D^*$. If another vertex w is also only dominated by z and x_A from $D' \cup D^*$, then such a vertex cannot be M_0-unmatched because the set of M_0-unmatched vertices forms an independent set. But then w is dominated by x_A, z and its M_0-matched neighbor, a contradiction. Hence, if z is M_0-unmatched, then x'_A is located by $D' \cup D^*$.

Suppose that z is not an M_0-unmatched vertex. Thus, z belongs to another bad (D_0, M_0)-matched 4-cycle B of $\overrightarrow{G}(S_{u,v})$ where $z = u_B$ and where A is dependent on B via x'_A (as illustrated in Fig. 2). If $z \notin D'$, then both z and x'_A are 1-dominated by $D' \cup D^*$ and hence are located by $D' \cup D^*$. Finally, if $z \in D'$, then x'_A is only dominated by z and x_A from $D' \cup D^*$. Suppose to the contrary that some other vertex w is also only dominated by z and x_A from $D' \cup D^*$. Then, w must be the neighbor of z in B that was removed from D_0, namely the vertex $w = u'_B$ (recall that $z = u_B$). Thus, u'_B is adjacent to x_A. If $A = R$, then $x_A \in \{u, v\}$ and x_A would be 3-dominated by D_0, a contradiction. Hence, $A \neq R$. If $x = v$, then we contradict the fact that v_A is adjacent to exactly two vertices of D_0, namely to u'_A and v'_A, and therefore could not be adjacent to $u'_B \in D$. Hence, $x = u$. But then B would be dependent on A via $w = u'_B$. However, recall that A is dependent on B via u'_A, implying that $\overrightarrow{G}(S_{u,v})$ would contain a 2-cycle joining A and B, a contradiction. Hence, if z is not an M_0-unmatched vertex, then once again x'_A is located by $D' \cup D^*$. This completes the proof of Claim B. ⋄

By Claim B, the set of D'-bad pairs is a proper subset of the set of D_0-bad pairs, implying that all remaining D'-bad pairs are associated with a pair of vertices of G^* neither of which belongs to the set D^*.

Step 2: We next consider all remaining D'-bad pairs associated with a pair of vertices of G^* neither of which belongs to the set D^*. For each such D'-bad pair $\{u, v\}$, we have $f(u, v) = \{u'', v''\}$ where $\{u'', v''\} \subseteq V^* \setminus D^*$. Let $\mathcal{C}$ be a component of $G^* - D^*$ that contains at least one edge. As observed earlier, $G^* - D^*$ has maximum degree at most 2. Thus, $\mathcal{C}$ is a path or a cycle. If $\mathcal{C}$ is a path, let $\mathcal{C}$ be given by $c_0c_1 \ldots c_{k-1}$, while if $\mathcal{C}$ is a cycle, let $\mathcal{C}$ be given by $c_0c_1 \ldots c_{k-1}c_0$ (possibly, $\mathcal{C}$ is a 2-cycle). We now consider an edge $c_ic_{(i+1) \bmod k}$ in $\mathcal{C}$, where $i \in \{0, \ldots, k-2\}$ if $\mathcal{C}$ is a path and where $i \in \{0, \ldots, k-1\}$ if $\mathcal{C}$ is a cycle. Let $\{u, v\}$ be a D_0-bad pair such that $f(u, v) = \{c_i, c_{(i+1) \bmod k}\}$, and let B be the bad (D_0, M_0)-matched 4-cycle of $S_{u,v}$ such that one of c_i and $c_{(i+1) \bmod k}$ is adjacent to u'_B and the other to v'_B. Let c_i be the neighbor of x'_B, where $x_B \in \{u_B, v_B\}$. We now propagate modifications of D' along a path in $\overrightarrow{G}(S_{u,v})$ in the same way as we did in Step 1, except that we start the modifications of D' along the oriented tree by replacing x'_B with x_B and then continuing exactly as before. The resulting modification of D' ensures that for every vertex in $\mathcal{C}$, at most one of its neighbors is removed from D_0. This process is done for all D_0-bad pairs associated to a pair of vertices of G^* neither of which belongs to the set D^*. Let D'' be the resulting modified set D_0.

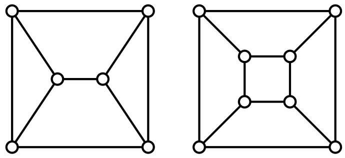
Claim C. *No D_0-bad pair is a $(D'' \cup D^*)$-bad pair. Further, the set of $(D'' \cup D^*)$-bad pairs is a proper subset of the set of $(D' \cup D^*)$-bad pairs.*

Proof of Claim C. It suffices to check as before the pairs of vertices that could possibly have been affected by the exchange arguments; that is, all vertices previously dominated by a vertex that has been removed from D' to construct D'' as well as all vertices removed from D' to construct D''. The proof is the same as in the proof of Claim B, except for the vertices in $G^* - D^*$. We therefore only prove that vertices that belong to components of $G^* - D^*$ that contain at least one edge are located by $D'' \cup D^*$. Let c be such a vertex in $G^* - D^*$. As observed earlier, such a vertex c belongs to either a path component or a cycle component of $G^* - D^*$. Further, the modifications of D' when constructing D'' ensure that for every vertex in $G^* - D^*$, at most one of its neighbors is removed from D_0.

We show next that c was 3-dominated by D_0. Suppose to the contrary that the vertex c is not 3-dominated by D_0 and therefore, by definition of $\mathcal{D}(M)$, is adjacent to both ends of some edge pq of M_0. In this case, since c has degree at least 1 in $G^* - D^*$ and therefore degree at least 2 in G^*, the edge pq must be an edge $x_Bx'_B$, where $x_B \in \{u_B, v_B\}$, in a bad (D_0, M_0)-matched 4-cycle B of $S_{u,v}$ for some D_0-bad pair $\{u, v\}$. However by Claim A(c), the vertex x'_B belongs to X. Therefore the vertex x_B, which is M_0-matched to x'_B, belongs to an odd component of $G - X$ that contains no M_0-unmatched vertex. However, the M_0-unmatched vertex c, which is adjacent to x_B, belongs to the same component of $G - X$ as x_B, a contradiction. Hence, the vertex c was 3-dominated by D_0.

We show now that vertex c is located by $D'' \cup D^*$. If c is 3-dominated by $D'' \cup D^*$, then this follows from the twin-freeness of G. Hence we may assume that c is not 3-dominated by $D'' \cup D^*$. Since the vertex c was 3-dominated by D_0, and at most one of its neighbors is removed from D_0, this implies that the vertex c has exactly two neighbors in D'' (and no neighbors in D^* in G), and is therefore 2-dominated by $D'' \cup D^*$. Suppose to the contrary that there is a vertex w that is not located from c by $D'' \cup D^*$. Let d be a vertex in D^* that is adjacent to c in G^*. Then there exists a D_0-bad pair $\{u, v\}$ such that $f(u, v) = \{c, d\}$.

Let B be the bad (D_0, M_0)-matched 4-cycle of $S_{u,v}$ such that one of c and d is adjacent to u'_B and the other to v'_B. Let c be the neighbor of x'_B, where $x_B \in \{u_B, v_B\}$. By Step 1, we know that $x'_B \in D''$ and $y_B \in D''$. Therefore, the vertex w must be the



(a) $C_3 \square K_2$. (b) $C_4 \square K_2$.

Fig. 4. The prisms $C_3 \square K_2$ and $C_4 \square K_2$.

vertex x_B. Let z be the neighbor of c in D'' that is different from x'_B. Since the set of neighbors of c in D'' is a subset of the set of its neighbors in D_0, we note that $\{x'_B, z\} \subset D_0$. If $x_B = v_B$, then v_B would be 3-dominated by D_0, contradicting the fact that B is a bad (D_0, M_0)-matched 4-cycle. Similarly, if $x_B = u$, then u would be 3-dominated by D_0, a contradiction. Hence, $x_B = u_B$ and in $S_{u,v}$ there is a bad (D_0, M_0)-matched 4-cycle that depends on B via the vertex z. But then z has at least two neighbors apart from c and x_B, contradicting the fact that G is cubic. Therefore, the vertex c is located by $D'' \cup D^*$. This completes the proof of Claim C. $\diamond$

Claim C implies that there is no $(D'' \cup D^*)$-bad pair. Thus, the set $D'' \cup D^*$ is a locating-dominating set of G. Therefore,

$$\gamma_L(G) \le |D| + |D^*| \le \alpha'(G) + \gamma(G^*) \le \alpha'(G) + \frac{|V^*|}{2} \le \alpha'(G) + \frac{n - 2\alpha'(G)}{2} = \frac{n}{2}.$$

This completes the proof of Theorem 3. $\square$

3.3. Tight examples

We remark that the prisms $C_3 \square K_2$ and $C_4 \square K_2$ (shown in Fig. 4(a) and (b), respectively) have location-domination number exactly one-half their order. However, it remains as an open problem to characterize all twin-free, cubic graphs G of order n that satisfy $\gamma_L(G) = \frac{n}{2}$. Note that the prisms $C_k \square K_2$ for $k \ge 5$ do not belong to this family.

4. Conclusion

We conclude the paper with several intriguing open problems and questions that we have yet to solve.

Problem 1. Characterize the extremal graphs that achieve equality in the bound of Theorem 3; that is, characterize the connected twin-free, cubic graphs having location-domination number exactly one-half their order.

Problem 2. Determine whether the result of Theorem 3 can be strengthened by proving Conjecture 2 for *subcubic* graphs.

Problem 3. Determine whether Theorem 3 can be extended to connected cubic graphs in general (allowing twins) with the exception of a finite set of forbidden graphs. Two such forbidden graphs are the complete graph K_4 and the complete bipartite graph $K_{3,3}$, but it is possible that these are the only two exceptions. Proving this would still be weaker than proving the conjecture of Henning and Löwenstein [9] that every cubic graph different from K_4 and $K_{3,3}$ has a *total* locating-dominating set of size at most one-half its order.

Problem 4. Determine whether every connected twin-free, cubic graph G satisfies $\gamma_L(G) \le \alpha'(G)$. More generally, determine classes of twin-free graphs G satisfying $\gamma_L(G) \le \alpha'(G)$. We remark that Garijo et al. [6] proved that every nontrivial twin-free graph G without 4-cycles satisfies $\gamma_L(G) \le \alpha'(G)$, and therefore Conjecture 2 holds for these graphs.

Acknowledgments

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