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Note

On the structure of arbitrarily partitionable graphs with given connectivity☆

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ABSTRACT

A graph $G = (V, E)$ is *arbitrarily partitionable* if for any sequence τ of positive integers adding up to $|V|$, there is a sequence of vertex-disjoint subsets of V whose orders are given by τ, and which induce connected subgraphs. Such a graph models, e.g., a computer network which may be arbitrarily partitioned into connected subnetworks. In this paper we study the structure of such graphs and prove that unlike in some related problems, arbitrarily partitionable graphs may have arbitrarily many components after removing a cutset of a given size ≥ 2. The sizes of these components grow exponentially, though.

1. Introduction

1.1. Arbitrarily partitionable graphs

Consider a computer network which we want to partition into disjoint, but *connected*, subnetworks of given sizes. If it is always feasible regardless of the sizes of the subnetworks, then the underlying graph, where computers are represented by vertices and links between two computers by edges, is *arbitrarily partitionable*.

More formally, let $n, \tau_1, \ldots, \tau_k$ be positive integers such that $\tau_1 + \cdots + \tau_k = n$. Then $\tau = (\tau_1, \ldots, \tau_k)$ is called a *decomposition* of n.

Let $G = (V, E)$ be a graph of order n and S a subset of V. By $G[S]$ we denote the subgraph of G induced by S.

Let $\tau = (\tau_1, \ldots, \tau_k)$ be a decomposition of n. The graph G is called τ*-partitionable* iff there exists a partition of V: $V_1, \ldots, V_k$ such that for each i, $1 \leq i \leq k$, $|V_i| = \tau_i$ and $G[V_i]$ is connected. In this case, τ is said to be *realizable* in G and $(V_1, \ldots, V_k)$ is a *realization* of τ in G.

A graph G of order n is *arbitrarily partitionable* (*AP* for short) iff for each decomposition τ of n, G is τ-partitionable.

1.2. On-line and recursive partitions

The problem of arbitrary partitionability gave rise to a list of natural stronger properties. Suppose for instance that the whole list of sizes of subnetworks is initially not known, but its elements are requested on-line, i.e., one by one. Using the graph modeling, this means that upon (any) request we must be able to provide a connected subgraph of a given order such

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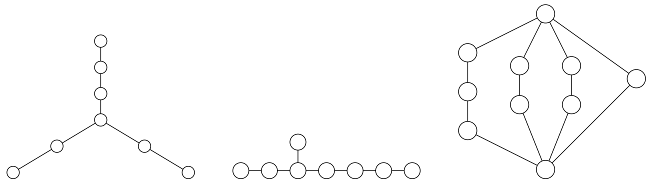

(a) $T_3(3, 2, 2)$. (b) $Cat(3, 5)$. (c) $B(3, 2, 2, 1)$.

Fig. 1. Examples of special graphs.

Table 1
Values of a, b ($b \geq a$) for which $Cat(a, b)$ is OLAP.

a	b	a	b	a	b
2	$\equiv 1 \pmod 2$	5	6, 7, 9, 11, 14, 19	8	11, 19
3	$\equiv 1, 2 \pmod 3$	6	$\equiv 1, 5 \pmod 6$	9, 10	11
4	$\equiv 1 \pmod 2$	7	8, 9, 11, 13, 15	11	12

that the remaining part of the graph retains the same feature. Graphs which have this property for any sequence of requests are called *on-line arbitrarily partitionable* (or *OLAP* for short).

In other words, a connected graph $G = (V, E)$ of order n is *on-line arbitrarily partitionable* iff for each integer $1 \leq \lambda \leq n-1$, there exists a subset V_λ of V such that $|V_\lambda| = \lambda$, $G[V_\lambda]$ is connected and $G[V \backslash V_\lambda]$ is OLAP. See [4] for details.

Another family of arbitrarily partitionable graphs has been considered in [3]. These were the *recursively arbitrarily partitionable* graphs. In this case we want not only to provide connected subgraphs, but also require that these subgraphs are themselves partitionable.

A graph $G = (V, E)$ of order n is called *recursively arbitrarily partitionable* (*RAP* for short) iff

- $G = K_1$ or
- G is connected and for each decomposition $\tau = (\tau_1, \ldots, \tau_k)$ of n, $k \geq 2$, there exists a partition of V: $V_1, \ldots, V_k$ such that for all i, $1 \leq i \leq k$, $|V_i| = \tau_i$ and $G[V_i]$ is RAP.

In [3], it has been shown that for every graph G, G is RAP $\Rightarrow$ G is OLAP $\Rightarrow$ G is AP, and that there exist AP graphs that are not OLAP and OLAP graphs that are not RAP.

1.3. Previous results

Since every graph containing a spanning AP graph is itself AP, much work have been done to investigate the 'simplest' potential (connected) spanning subgraphs, i.e., trees, which are 1-connected. Below we recall a number of previous results, which, as we shall argue in the following section, provide much insight into the structure of 1- and 2-connected AP graphs, and in particular into the number of components left after removal of a (minimal) cutset, and the sizes of these components.

The following is the central result among these. It provides an upper bound on the degree in AP trees (and thus on the number of components left in an AP 1-connected graph after removing a cut-vertex).

Theorem 1 (*[1]*)**.** *If a tree T is AP, then its maximum degree is at most* 4*. Moreover, every vertex of degree* 4 *in T is adjacent to a leaf.*

In [4,3], OLAP and RAP-trees have been completely characterized. To recall these characterizations, we need the following notations:

- A *k-pode* $T_k(t_1, \ldots, t_k)$ is a tree of order $1 + \sum_{i=1}^{k} t_i$ composed of k paths of respective orders $t_1, \ldots, t_k$, connected to a unique node, called the *root* of the k-pode (cf. Fig. 1(a)).
- Let a and b be two positive integers. A caterpillar $Cat(a, b)$ is a tree of order $a+b$ composed of three paths of order a, b and 2 sharing exactly one node, called the *root* of the caterpillar. $Cat(a, b)$ is isomorphic to $T_3(a-1, b-1, 1)$ (cf. Fig. 1(b)).

Theorem 2 (*[4]*)**.** *A tree T is OLAP if and only if either T is a path or T is a caterpillar $Cat(a, b)$ with a and b given in* Table 1 *or T is the* 3*-pode* $T_3(2, 4, 6)$.

Theorem 3 (*[3]*)**.** *A tree T is RAP if and only if either T is a path or T is a caterpillar $Cat(a, b)$ with a and b given in* Table 2 *or T is the* 3*-pode* $T_3(2, 4, 6)$.

In terms of 2-connected graphs, let us consider the 'simplest' of such graphs forming the family of so called *balloons*. Let $b_1, \ldots, b_k$ be positive integers, $k \geq 2$. A *k-balloon* $B(b_1, \ldots, b_k)$ is a graph of order $2 + \sum_{i=1}^{k} b_i$ composed of two vertices (called *roots*) linked by k paths (called *branches*) of *widths* (the numbers of internal vertices) $b_1, \ldots, b_k$ (cf. Fig. 1(c)).

Table 2
Values of a, b ($b \geq a$) for which $Cat(a, b)$ is RAP.

a	b	a	b	a	b
2	$\equiv 1 \pmod 2$	4	$\equiv 1 \pmod 2$	6	7
3	$\equiv 1, 2 \pmod 3$	5	6, 7, 9, 11, 14, 19	7	8, 9, 11, 13, 15

Theorem 4 ([3]). *If a k-balloon is RAP, then $k \leq 5$. This bound is tight.*

This result has been extended to OLAP k-balloons:

Theorem 5 ([2]). *If a k-balloon is OLAP, then $k \leq 5$.*

Upper bounds for the size of the smallest branch of a RAP or OLAP k-balloon have also been given:

Theorem 6 ([2]). *Let $B(b_1, \ldots, b_k)$ be a k-balloon with $k \geq 4$ and $b_1 \leq \cdots \leq b_k$. If $B(b_1, \ldots, b_k)$ is OLAP, then $b_1 \leq 11$. If $B(b_1, \ldots, b_k)$ is RAP, then $b_1 \leq 7$.*

2. Size and number of components after removing a cutset of size at most 2

Observation 7. If G contains a spanning subgraph which is AP (resp. OLAP, RAP), then G is AP (resp. OLAP, RAP). In particular, if G is traceable (contains a Hamiltonian path), then G is RAP (and thus also OLAP and AP).

This simple remark suggests the following straightforward generalization. Suppose that G is an AP (resp. OLAP, RAP) graph containing a one- or two-element cutset S. Then we may construct of it another graph which is the more AP (resp. OLAP, RAP), simply by replacing each component of $G - S$ with a path of the corresponding order.

Observation 8. Let $G = (V, E)$ be a graph with a cutset S, and let $V_1, \ldots, V_k$ be the components of $G[V \backslash S]$:

- if $|S| = 1$ and G is AP (resp. OLAP, RAP), then the k-pode $T_k(|V_1|, \ldots, |V_k|)$ is AP (resp. OLAP, RAP);
- if $|S| = 2$ and G is AP (resp. OLAP, RAP), then the k-balloon $B(|V_1|, \ldots, |V_k|)$ is AP (resp. OLAP, RAP).

Observation 8 and Theorems 1–6 yield the following summary concerning all 1- and 2-connected AP graphs.

Corollary 9. *Let G be a graph with a cutset S, and let $V_1, \ldots, V_k$ the components of $G[V \backslash S]$ with $|V_1| \leq \cdots \leq |V_k|$:*

- *if $|S| = 1$, then*
  - *if G is AP, then $k \leq 4$ and if $k = 4$, then $|V_1| = 1$;*
  - *if G is OLAP (resp. RAP), then $k \leq 3$, and if $k = 3$, then either $(|V_1|, |V_2|, |V_3|) = (1, a-1, b-1)$ with values a and b given in Table 1 (resp. 2), or $(|V_1|, |V_2|, |V_3|) = (2, 4, 6)$;*
- *if $|S| = 2$, then*
  - *if G is OLAP, then $k \leq 5$;*
  - *if G is OLAP (resp. RAP) and $k \in \{4, 5\}$, then $|V_1| \leq 11$ (resp. ≤ 7).*

3. Number of components after removing a cutset of size $k \geq 2$ in AP graphs

In the previous section, we argued that if we remove a cutset of size 1 from a (1-connected) graph G, then the number of remaining components is at most 4 if G is AP, and 3 if G is OLAP or RAP. A similar result on the bounded number of components extends to the case of removal of a cutset of size 2 from a (1- or 2-connected) OLAP or RAP graph, when this number is at most 5. Surprisingly, the same cannot be generalized for AP graphs.

In this section, we will prove that for any size $k \geq 2$ of a cutset, a similar statement does not hold for AP graphs.

Theorem 10. *For any integers $k \geq 2$ and $c \geq 2$, there exists an AP graph $G = (V, E)$ of connectivity k such that $G[V \backslash S]$ consists of exactly c components for every k-element cutset S of G.*

Proof. We shall present a construction of such a graph G for every pair of integers $k, c \geq 2$.

We first consider the case when $c \leq k$. Let $G = (S' \cup P' \cup S'' \cup P'', E)$ be the graph with $2k$ vertices constructed as follows:

- $G[S']$ and $G[S'']$ are both stable sets, each containing $c - 1$ vertices;
- $G[P']$ and $G[P'']$ are both paths with $k - c + 1$ vertices each;
- every vertex of $S' \cup P'$ is adjacent to all the vertices of $S'' \cup P''$.

Clearly, G is an AP graph, since it contains as a subgraph the complete bipartite graph $K_{k,k}$, which is Hamiltonian. By the same reason, G has connectivity k and contains exactly two cutsets of size k, i.e., $S' \cup P'$ and $S'' \cup P''$. After removing any of these, we obtain exactly c components, i.e., a path with $k - c + 1 \geq 1$ vertices and $c - 1$ isolated vertices.

Now, we assume that $c > k$.

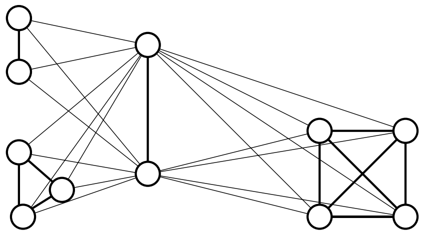

Fig. 2. Graph $K_2(2, 3, 4)$.

We denote by $K_k(a_1, \ldots, a_c)$ the graph formed of $c + 1$ cliques, one of size k, the others of size $a_1, \ldots a_c$, by adding all the edges between the vertices of the clique of size k and the vertices of all other cliques (see Fig. 2). Clearly G is k-connected and the vertices of the clique K_k form a unique cutset of size k in $K_k(a_1, \ldots, a_c)$.

Let G be any graph $K_k(a_1, \ldots, a_c)$ with values $a_1, \ldots, a_c$ chosen (consecutively) as follows:

1. $1 \le a_1 \le \cdots \le a_c$;
2. for any i, $1 \le i \le c$, we denote $n_i = 1 + \sum_{1 \le j \le i} a_j$;
3. for any i, $1 \le i \le c - 1$, choose a_{i+1} such that
   (a) $\forall j, 2 \le j \le n_i, a_{i+1} \equiv 0 \pmod j$,
   (b) $a_{i+1} \ge n_i a_i$.
   For example, $a_i \cdot n_i!$ is a possible value for a_{i+1}.

Let $\tau = (\tau_1, \ldots, \tau_l)$ be any decomposition of $n = k + \sum_{1 \le i \le c} a_i$, with $\tau_1 \le \cdots \le \tau_l$. To show that τ is realizable in G we consider two cases.

First case: $\tau_l \ge a_{c-1}$.

In that case, $\tau_l \ge n_{c-2} = 1 + \sum_{1 \le i \le c-2} a_i$. Thus, the part (vertex subset) of size τ_l may be chosen so that it 'covers' all the cliques K_{a_i} with $i \le c - 2$, plus one of the vertices of K_k and possibly some vertices of the cliques $K_{a_{c-1}}$ and K_{a_c}. The remaining graph is induced by the rest of the vertices from K_k, $K_{a_{c-1}}$ and K_{a_c}, and is obviously traceable.

Second case: $\tau_l < a_{c-1}$.

For each i, $1 \le i \le \tau_l$, we denote by q_i the number of terms of τ having value i.

We thus have $n = \sum_{1 \le i \le \tau_l} i \cdot q_i$.

Then there exists an integer α, $1 \le \alpha \le \tau_l$, such that $\alpha \cdot q_\alpha \ge \frac{n}{\tau_l}$. Thus, $\alpha \cdot q_\alpha > \frac{n}{a_{c-1}} > \frac{a_c}{a_{c-1}} \ge \frac{n_{c-1} a_{c-1}}{a_{c-1}} = n_{c-1}$.

We denote by s the integer such that for all $i \le s$, $a_i \not\equiv 0 \pmod \alpha$ and for all $i > s$, $a_i \equiv 0 \pmod \alpha$. Its existence is guaranteed by property 3a. Note that s may in particular be equal to 0.

Because for each $i > s$, $a_i \equiv 0 \pmod \alpha$, and $\alpha \cdot q_\alpha > n_{c-1}$, we may cover the cliques $K_{a_{s+1}}, \ldots, K_{a_{c-1}}$ with parts of size α.

If $s \ge 2$, since $a_s \not\equiv 0 \pmod \alpha$, we have $\alpha > n_{s-1}$ by property 3a. On the other hand, $\alpha \le a_c$. It means that we may choose one part of size α so that it covers all the cliques $K_{a_1}, \ldots, K_{a_{s-1}}$ plus one vertex of K_k and possibly some vertices of K_{a_c}.

Thus the remaining graph is induced by

- the vertices of K_{a_c} and K_k if $s = 0$,
- the vertices of K_{a_c}, K_k and K_{a_1} if s = 1,
- the remaining vertices of K_{a_c}, $k - 1$ vertices of K_k and the vertices of K_{a_s} if $s \ge 2$.

In every case, such a graph is again traceable. □

Balloons. The previous result (Theorem 10) can be adapted to the special case of balloons. The benefit from such a modification is that it gives (for the case $k = 2$) examples of graphs with a linear number of edges (with respect to n – the order of a graph), in contrast to the examples presented above, where the number of edges may be quadratic.

Theorem 11. *For any $k \ge 1$, there exists an AP k-balloon.*

Proof. We consider a k-balloon $B(b_1, \ldots, b_k)$ where branches have the same size as the cliques of $K_k(a_1, \ldots, a_c)$ given in the proof of Theorem 10, i.e., for $b_1 \le \cdots \le b_k$ we denote $n_i = 1 + \sum_{1 \le j \le i} b_j$, and choose b_i as follows:

- $b_1 \ge 1$;
- for any $i \le k - 1$, choose b_{i+1} such that:
  1. $\forall j, 2 \le j \le n_i, b_{i+1} \equiv 0 \pmod j$,
  2. $b_{i+1} \ge n_i b_i$.

Using the same argument as the one used for $K_k(a_1, \ldots, a_c)$, it is easy to see that such a balloon is AP. □

4. Size of components

In this section, we will show that, though the number of components after removing a cutset of size at least 2 from an AP graph may be arbitrarily large, then the size of these components must grow exponentially with their number.

Theorem 12. *Let $G = (V, E)$ be an AP graph with n vertices, S a cutset of G of size k, $c_1, \dots, c_l$ the orders of the components of $G[V \setminus S]$, where $l > k$ and $1 \le c_1 \le c_2 \le \cdots \le c_l$. Then the values of the sequence $(c_i)_{i \ge 1}$ grow exponentially with i.*

To prove this theorem, we use Lemma 13 to 15:

Lemma 13. *Let $G = (V, E)$ be a graph with n vertices, S a cutset of G of size k, $c_1, \dots, c_l$ the orders of the components of $G[V \setminus S]$, where $l > k$ and $1 \le c_1 \le c_2 \le \cdots \le c_l$.*

Let $a, q_1, \dots, q_l, r_1, \dots, r_l$ be non-negative integers such that:

- $2 \le a \le n - 1$;
- *for any i, $1 \le i \le l$, $c_i = q_i a + r_i$ with $r_i < a$.*

If G is AP, then

$$\sum_{1 \le i \le l} r_i \le (k + 1) \cdot (a - 1).$$

Proof. Let $G_1, G_2, \dots, G_l$ be the components of $G[V \setminus S]$ of size $c_1, c_2, \dots, c_l$, respectively. Consider a decomposition $\tau = (a, \dots, a, r)$ of n with $r < a$, and any of its realizations in G. Now suppose we remove from G the vertices of all parts (in the realization) of size a each of which is contained entirely in one of the subgraphs $G_1, \dots, G_l$. We thus must have at least $\sum_{1 \le i \le k} r_i + k$ vertices in the remaining graph. On the other hand, every part (in the realization) left in our graph must contain at least one of k vertices of the cutset S or has size different from a (and there is only one such part in the graph). Therefore, the remaining graph is induced by at most $k \cdot a + r \le k \cdot a + a - 1$ vertices. Combining the two observations, we obtain that $\sum_{1 \le i \le k} r_i + k \le k \cdot a + a - 1$, and the thesis follows. □

Corollary 14. *Let $G = (V, E)$ be a graph with n vertices, S a cutset of G of size k, $c_1, \dots, c_l$ the orders of the components of $G[V \setminus S]$, where $l > k$ and $1 \le c_1 \le c_2 \le \cdots \le c_l$. If G is AP, then for any i, $2 \le i \le l$,*

$$c_i \ge \frac{1}{k} \sum_{1 \le j < i} c_j.$$

Proof. For any fixed $i \le l$, let us apply Lemma 13 with $a = c_i + 1$. Then for all $j \le i$, we have $r_j = c_j$. Thus, by Lemma 13, $\sum_{1 \le j < i} c_j + c_i + \sum_{i < j \le l} r_j \le (k + 1) \cdot c_i$. Since $\sum_{i < j \le l} r_j \ge 0$, we obtain the thesis. □

The following lemma completes the proof of Theorem 12:

Lemma 15. *If the assumptions of Theorem 12 hold, then:*

$$\forall i \ge 2, \quad c_i \ge \left(1 + \frac{1}{k}\right)^{i-2} \times \frac{c_1}{k}.$$

Proof. Consider the sequence $(v_i)_{i \ge 1}$ defined by $v_1 = c_1$ and for all $i \ge 2$, $v_i = \frac{1}{k} \sum_{1 \le j < i} v_j$. Corollary 14 implies that for any $i \ge 1$, $c_i \ge v_i$.

We have $v_2 = \frac{v_1}{k} = \frac{c_1}{k}$ and $v_3 = \frac{1}{k}(v_1 + v_2) = v_2 + \frac{1}{k} v_2 = \left(1 + \frac{1}{k}\right) v_2$.

For each integer $i \ge 3$, $v_{i+1} = \frac{1}{k} \sum_{1 \le j \le i} v_j = \frac{1}{k} \left(v_i + \sum_{1 \le j \le i-1} v_j\right) = \frac{1}{k} v_i + v_i = \left(1 + \frac{1}{k}\right) v_i$.

Thus, by induction, we have $v_{i+1} = \left(1 + \frac{1}{k}\right)^{i-1} v_2 = \left(1 + \frac{1}{k}\right)^{i-1} \frac{c_1}{k}$. □

Note that, even if the lower bound given in the proof of Theorem 12 is exponential, there remains a large gap between this bound and the order of the example used to prove Theorem 10 (second case). Thus, it would be interesting to find smaller examples or to improve the lower bound.

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