Smallest C_{2l+1}-Critical Graphs of Odd-Girth $2k+1$

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Abstract. Given a graph H, a graph G is called H-critical if G does not admit a homomorphism to H, but any proper subgraph of G does. Observe that K_{k-1}-critical graphs are the classic k-(colour)-critical graphs. This work is a first step towards extending questions of extremal nature from k-critical graphs to H-critical graphs. Besides complete graphs, the next classic case is odd cycles. Thus, given integers $l \geq k$ we ask: what is the smallest order $\eta(k,l)$ of a C_{2l+1}-critical graph of odd-girth at least $2k+1$? Denoting this value by $\eta(k,l)$, we show that $\eta(k,l) = 4k$ for $l \leq k \leq \frac{3l+i-3}{2}$ ($2k = i \bmod 3$) and that $\eta(3,2) = 15$. The latter is to say that a smallest graph of odd-girth 7 not admitting a homomorphism to the 5-cycle is of order 15 (there are at least 10 such graphs on 15 vertices).

1 Introduction

A k-critical graph is a graph which is k-chromatic but any proper subgraph of it is $(k-1)$-colourable. Extremal questions on critical graphs are a rich source of research in graph theory. Many well-known results and conjectures are about this subject, see for example [4,6–8,15]. Typical questions are for example:

Problem 1. What is the smallest possible order of a k-critical graph having a certain property, such as low clique number, high girth or high odd-girth?

For example, Erdős' proof of existence of graphs of high girth and high chromatic number [5] is a starting point for Problem 1. This fact implies that each of the above questions have a finite answer. The specific question of the smallest 4-critical graph without a triangle has received considerable attention: Grötzsch built a graph on 11 vertices which is triangle-free and not 3-colourable.

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Harary [11] showed that any such graph must have at least 11 vertices, and Chvátal [3] showed that the Grötzsch graph is the only one with 11 vertices.

Every graph with no odd cycle being 2-colourable, in the context of colouring, it is of interest to consider the *odd-girth*, the size of a smallest odd-cycle of a graph (rather than the girth). Extending construction of Grötzsch's graph, Mycielski [14] introduced the construction, now knows as the Mycielski construction, to increase the chromatic number without increasing the clique number. A generalization of this construction is used to build 4-critical graphs of high odd-girth, more precisely every *generalized Mycielski construction on* C_{2k+1}, denoted $M_k(C_{2k+1})$ is a 4-critical graph. The graph $M_2(C_5)$ is simply the classic Mycielski construction for C_5, that is, the Grötzsch graph. $M_k(C_{2k+1})$ has odd-girth $2k+1$, and several authors (starting with Payan [18]) showed that $M_k(C_{2k+1})$ is 4-chromatic for any $k \geq 1$ and in fact 4-critical, thus providing an upper bound of $2k^2+k+1$ for the minimum order of a 4-critical graph of odd-girth at least $2k+1$. We refer to [10,16,19,20] for several other proofs. Among these authors, Ngoc and Tuza [16] asked whether this upper bound of $2k^2+k+1$ is essentially optimal. The best known lower bound is due to Jiang [13], who proved the bound of $(k-1)^2+2$, which establishes the correct order of magnitude at $\Theta(k^2)$ (see [17] for an earlier but weaker lower bound).

The current work is a first step towards generalizing these extremal questions for k-critical graphs to H-critical graphs, defined using the terminology of homomorphisms. A *homomorphism* of a graph G to a graph H is a vertex-mapping that preserves adjacency, i.e., a mapping $\psi : V(G) \to V(H)$ such that if x and y are adjacent in G, then $\psi(x)$ and $\psi(y)$ are adjacent in H. If there exists a homomorphism of G to H, we may write $G \to H$ and we may say that G is H-colourable. In the study of homomorphisms, it is usual to work with the *core* of a graph, that is, a minimal subgraph which admits a homomorphism from the graph itself. It is not difficult to show that a core of any graph is unique up to isomorphism. A graph is said to be *a core* if it admits no homomorphism to a proper subgraph. We refer to the book [12] for a reference on these notions.

It is a classic fact that homomorphisms generalize proper vertex-colourings. Indeed a homomorphism of G to K_k is equivalent to a k-colouring of G. However, the extension of the notion of colour-criticality to a homomorphism-based one has been almost forgotten. As defined by Catlin [2], for a graph H, (we may assume H is a core), a graph G is said to be *H-critical* if G does not have a homomorphism to H but any proper subgraph of G does. Thus:

Observation 2. *A graph G is k-critical if and only if it is K_{k-1}-critical.*

This gives a large number of interesting extremal questions. By Observation 2, these questions are well-studied when H is a complete graph. The next most important family to be considered is when H is an odd cycle. This is the goal of this work. More precisely we ask:

Problem 3. Given positive integers k, l, what is the smallest order $\eta(k,l)$ of a C_{2l+1}-critical graph of odd-girth at least $2k+1$?

In this work, we study Problem 3 when $l \geq 2$. As we discuss in Sect. 2, it follows from a theorem of Gerards [9] that $\eta(k,l) \geq 4k$ whenever $l \leq k$, and $\eta(k,k) = 4k$. We prove (in Sect. 3) that, surprisingly, $\eta(k,l) = 4k$ whenever $l \leq k \leq \frac{3l+i-3}{2}$ (with $2k = i$ mod 3). We then prove (in Sect. 4) that $\eta(3,2) = 15$. We conclude with further research questions in the last section. Table 1 summarizes the known bounds for Problem 3 and small values of k and l.

Note that the value of $\eta(3,2)$ indeed was the initial motivation of this work. In [1], we use the fact that $\eta(3,2) = 15$ to prove that if a graph B of odd-girth 7 has the property that any series-parallel graph of odd-girth 7 admits a homomorphism to B, then B has at least 15 vertices.

Table 1. Known values/bounds on the smallest order of a not C_{2l+1}-colourable graph of odd-girth $2k+1$. Bold values are proved in this paper.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$l=1$	4	11[11]	**15**–22 [Th. 12]–[18]	**17**–37 [Co. 13]–[18]	**20**–56 [Co. 8]–[18]	27–79 [13]–[18]	38–106 [13]–[18]	51–137 [13]–[18]
$l=2$	3	**8** [Co. 6]	**15** [Th. 12]	**17**–37 [Co. 13]–[18]	**20**–56 [Co. 8]–[18]	**24**–79 [Co. 8]–[18]	**28**–106 [Co. 8]–[18]	**32**–137 [Co. 8]–[18]
$l=3$	3	5	**12** [Co. 6]	**16** [Th. 10]	**20**–56 [Co. 8]–[18]	**24**–79 [Co. 8]–[18]	**28**–106 [Co. 8]–[18]	**32**–137 [Co. 8]–[18]
$l=4$	3	5	7	**16** [Co. 6]	**20** [Th. 10]	**24**–79 [Co. 8]–[18]	**28**–106 [Co. 8]–[18]	**32**–137 [Co. 8]–[18]
$l=5$	3	5	7	9	**20** [Co. 6]	**24** [Th. 10]	**28** [Th. 10]	**32**–137 [Co. 8]–[18]
$l=6$	3	5	7	9	11	**24** [Co. 6]	**28** [Th. 10]	**32** [Th. 10]
$l=7$	3	5	7	9	11	13	**28** [Co. 6]	**32** [Th. 10]
$l=8$	3	5	7	9	11	13	15	**32** [Co. 6]

2 Preliminaries

This section is devoted to introduce useful preliminary notions and results.

Circular chromatic number. We recall some basic notions related to circular colourings. For a survey on the matter, consult [21]. Given two integers p and q with $gcd(p,q) = 1$, the *circular clique* $C(p,q)$ is the graph on vertex set $\{0,\ldots,p-1\}$ with i adjacent to j if and only if $q \leq |i-j| \leq p-q$. A homomorphism of a graph G to $C(p,q)$ is called a (p,q)-colouring, and the *circular chromatic number* of G, denoted $\chi_c(G)$, is the smallest rational p/q such that G has a (p,q)-colouring. Since $C(p,1)$ is the complete graph K_p, we have $\chi_c(G) \leq \chi(G)$. On the other hand $C(2l+1,l)$ is the cycle C_{2l+1}. Thus C_{2l+1}-colourability is about deciding whether $\chi_c(G) \leq 2+\frac{1}{l}$. It is a well-known fact that $C(p,q) \to C(r,s)$ if and only if $\frac{p}{q} \leq \frac{r}{s}$ (e.g. see [21]), in particular we will use the fact that $C(12,5) \to C_5$.

Odd-K_4's and a theorem of Gerards. The following notion will be central in our proofs. An *odd-K_4* is a subdivision of the complete graph K_4 where each of the four triangles of K_4 has become an odd-cycle [9]. Furthermore, we call it a $(2k+1)$-*odd-*K_4 if each such cycle has length exactly $2k+1$. Since subdivided triangles are the only odd-cycles of an odd-K_4, the odd-girth of a $(2k+1)$-odd-K_4 is $2k+1$. The following is an easy fact about odd-K_4's whose proof we leave as an exercise.

Proposition 4. *Let K be an odd-K_4 of odd-girth at least $2k+1$. Then, K has order at least $4k$, with equality if and only if K is a $(2k+1)$-odd-K_4. Furthermore, in the latter case any two disjoint edges of K_4 are subdivided the same number of times when constructing K.*

A $(2k+1)$-odd-K_4 is, more precisely, referred to as an (a,b,c)-*odd-*K_4 if three edges of a triangle of K_4 are subdivided into paths of length a, b and c respectively (by Proposition 4 this is true for all four triangles). Note that while the terms "odd-K_4" or "$(2k+1)$-odd-K_4" refer to many non-isomorphic graphs, an (a,b,c)-odd-K_4 $(a+b+c=2k+1)$ is unique up to a relabeling of vertices.

An *odd-*K_3^2 is a graph obtained from three disjoint odd-cycles and three disjoint paths (possibly of length 0) joining each pair of cycles [9]. Thus, in such graph, any two of the three cycles have at most one vertex in common (if the path joining them has length 0). Hence, an odd-K_3^2 of odd-girth at least $2k+1$ has order at least $6k$.

Theorem 5. (Gerards [9]). *If G has neither an odd-K_4 nor an odd-K_3^2 as a subgraph, then it admits a homomorphism to its shortest odd-cycle.*

Corollary 6. *For any positive integer k, we have $\eta(k,k)=4k$.*

Proof. Consider a C_{2k+1}-critical graph G of odd-girth $2k+1$. It follows from Theorem 5 that G contains either an odd-K_4, or an odd-K_3^2. If it contains the latter, then G has at least $6k$ vertices. Otherwise, G must contain an odd-K_4 of odd-girth at least $2k+1$, and then by Proposition 4, G has at least $4k$ vertices. This shows that $\eta(k,k)\geq 4k$.

Moreover, any $(2k+1)$-odd-K_4 has order $4k$ (by Proposition 4) and admits no homomorphism to C_{2k+1}, showing that $\eta(k,k)\leq 4k$. □

Since C_{2l+3} maps to C_{2l+1} and by transitivity of homomorphisms, a graph with no homomorphism to C_{2l+1} also has no homomorphism to C_{2l+3}. Thus:

Observation 7. *Let k,l be two positive integers. We have $\eta(k,l)\geq\eta(k,l+1)$.*

We obtain this immediate consequence of Corollary 6 and Observation 7:

Corollary 8. *For any two integers k,l with $k\geq l\geq 1$, we have $\eta(k,l)\geq 4k$.*

3 Rows of Table 1

In this section, we study the behavior of $\eta(k, l)$ when l is a fixed value, that is, the behavior of each row of Table 1. Note again that whenever $l \geq k+1$, $\eta(k, l) = 2k+1$. As mentioned before, the first row (i.e. $l = 1$) is about the smallest order of a 4-critical graph of odd-girth $2k+1$ and we know $\eta(k, 1) = \Theta(k^2)$ [17].

It is not difficult to observe that for $k \geq l$, the function $\eta(k, l)$ is strictly increasing, in fact with a little bit of effort we can even show the following.

Proposition 9. *For $k \geq l$, we have $\eta(k+1, l) \geq \eta(k, l) + 2$.*

Proof. Let G be a C_{2l+1}-critical graph of odd-girth $2k+3$ and order $\eta(k+1, l)$. Consider any $(2k+3)$-cycle $v_0 \cdots v_{2k+2}$ of G, and build a smaller graph by identifying v_0 with v_2 and v_1 with v_3. It is not difficult to check that the resulting graph has odd-girth exactly $2k+1$ and does not map to C_{2l+1} (otherwise, G would), proving the claim. □

While we expect that for a fixed l, $\eta(k, l)$ grows quadratically in terms of k, we show, somewhat surprisingly, that at least just after the threshold of $k = l$, the function $\eta(k, l)$ only increases by 4 when l increases by 1, implying that Proposition 9 cannot be improved much in this formulation. More precisely, we have the following theorem.

Theorem 10. *For any $k, l \geq 3$ and $l \leq k \leq \frac{3l+i-3}{2}$ (where $2k = i \bmod 3$), we have $\eta(k, l) = 4k$.*

To prove this theorem, we give a family of C_{2l+1}-critical odd-$K4$'s which are of odd-girth $2k+1$. This is done in the next theorem, after which we give a proof of Theorem 10.

Given a graph G, a *thread* of G is a path in G where the internal vertices have degree 2 in G. When G is clear from the context, we simply use the term thread.

Theorem 11. *Let $p \geq 3$ be an integer. If p is odd, any (a, b, c)-odd-K_4 with $(a, b, c) \in \{(p-1, p-1, p), (p, p, p)\}$ has no homomorphism to C_{2p+1}. If p is even, any $(p-1, p, p)$-odd-K_4 has no homomorphism to C_{2p+1}.*

Proof. Let $(a, b, c) \in \{(p-1, p-1, p), (p, p, p), (p-1, p, p)\}$ and let K be an (a, b, c)-odd-K_4. Let t, u, v, w be the vertices of degree 3 in K with the tu-thread of length a, the uv-thread of length b and the tv-thread of length c. We now distinguish two cases depending on the parity of p and the values of (a, b, c).

Case 1. Assume that p is odd and $(a, b, c) \in \{(p-1, p-1, p), (p, p, p)\}$. By contradiction, we assume that there is a homomorphism h of K to C_{2p+1}. Then, the cycle C_{tvw} formed by the union of the tv-thread, the vw-thread and the tw-thread is an odd-cycle of length $a+b+c$. Therefore, its mapping by h to C_{2p+1} must be *onto*. Thus, u has the same image by h as some vertex u' of C_{tvw}. Note that u' is not one of t, v or w, indeed by identifying u with any of these vertices we obtain a graph containing an odd-cycle of length p or $2p-1$; thus, this identification cannot be extended to a homomorphism to C_{2p+1}. Therefore, u' is an internal vertex of one of the three maximal threads in C_{tvw}. Let C_u be the odd-cycle of length $a+b+c$ containing u and u'. After identifying u and u', C_u is transformed into two cycles, one of them being odd. If $(a,b,c)=(p,p,p)$, then C_u has length $3p$. Then, the two newly created cycles have length at least $p+1$, and thus at most $2p-1$. If $(a,b,c)=(p-1,p-1,p)$, then C_u has length $3p-2$, and the two cycles have length at least p and at most $2p-2$. In both cases, we have created an odd-cycle of length at most $2p-1$. Hence, this identification cannot be extended to a homomorphism to C_{2p+1}, a contradiction.

Case 2. Assume that p is even and $(a,b,c)=(p-1,p,p)$, and that h is a homomorphism of K to C_{2p+1}. Again, the image of C_{tvw} by h is onto, and u has the same image as some vertex u' of C_{tvw}. If $u'=t$, identifying u and u' produces an odd $(p-1)$-cycle, a contradiction. If $u' \in \{v,w\}$, then we get a $(2p-1)$-cycle, a contradiction. Thus, u' is an internal vertex of one of the three maximal threads in C_{tvw}. Let C_u be the odd-cycle of length $3p-1$ containing u and u'. As in Case 1, after identifying u and u', C_u is transformed into two cycles, each of length at least p and at most $2p-1$; one of them is odd, a contradiction. □

We note that Theorem 11 is tight, in the sense that if p is odd and $(a,b,c) \in \{(p-1,p-1,p),(p,p,p)\}$ or if p is even and $(a,b,c)=(p-1,p,p)$, any (a,b,c)-odd-K_4 has a homomorphism to C_{2p-1}.

We can now prove Theorem 10.

Proof (Proof of Theorem 10). By Corollary 8, we know that $\eta(k,l) \geq 4k$. We now prove the upper bound. Recall that $\eta(k,l) \leq \eta(k,l-1)$. If $2k=0 \bmod 3$, then $p=\frac{2k+3}{3}$ is an odd integer, and $p \leq l$. By Theorem 11, a $(p-1,p-1,p)$-odd-K_4, which has order $6p-6=4k$, has no homomorphism to C_{2p+1}, and thus $\eta(k,l) \leq \eta(k,p) \leq 4k$. Similarly, if $2k=1 \bmod 3$, then $p=\frac{2k+2}{3}$ is an even integer, and $p \leq l$. By Theorem 11, a $(p-1,p,p)$-odd-K_4, which has order $6p-4=4k$, has no homomorphism to C_{2p+1}, and thus $\eta(k,l) \leq \eta(k,p) \leq 4k$. Finally, if $2k=2 \bmod 3$, then $p=\frac{2k+1}{3}$ is an odd integer, and $p \leq l$. By Theorem 11, a (p,p,p)-odd-K_4, which has order $6p-2=4k$, has no homomorphism to C_{2p+1}, and thus $\eta(k,l) \leq \eta(k,p) \leq 4k$. □

4 The Value of $\eta(3,2)$

We now determine $\eta(3,2)$, which is not covered by Theorem 11. By Corollary 8, we know that $\eta(3,2) \geq 12$. In fact, we will show that $\eta(3,2)=15$. Using a computer search, Gordon Royle (private communication, 2016) has found that there are at least ten graphs of order 15 and odd-girth 7 that do not admit a homomorphism to C_5. For example, see the three graphs of Fig. 1. Thus, $\eta(3,2) \leq 15$. Next, we prove that this upper bound is tight.

Theorem 12. *Any graph G of order at most 14 and odd-girth at least 7 admits a homomorphism to C_5, and thus $\eta(3,2) = 15$.*

Proof. We consider a C_5 on the vertex set $\{0,1,2,3,4\}$ where vertex i is adjacent to vertices $i+1$ and $i-1$ (modulo 5). Thus, in the following, to give a C_5-colouring we will give a colouring using elements of $\{0,1,2,3,4\}$ where adjacent pairs are mapped into (cyclically) consecutive elements of this set.

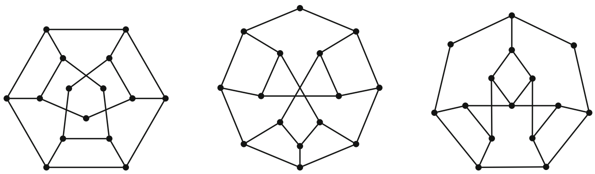

Fig. 1. Three C_5-critical graphs of order 15 and odd-girth 7.

Given a graph G and a vertex v of it, we partition $V(G)$ into four sets $\{v\}, N_1(v), N_2(v)$ and $N_{3^+}(v)$ where $N_1(v)$ (respectively $N_2(v)$) designates the set of vertices at distance exactly 1 (respectively 2) of v, and $N_{3^+}(v)$ the vertices at distance 3 or more of v. A proper 3-colouring of $G[N_{3^+}(v)]$ using colours c_1, c_2 and c_3 is said to be *v-special* if:

(i) each vertex with colour c_3 is an isolated vertex of $G[N_{3^+}(v)]$,
(ii) no vertex from $N_2(v)$ sees both colours c_1 and c_2.

A key observation is the following: given any graph G, if for some vertex v of G, there exists a v-special colouring of $G[N_{3^+}(v)]$, then G maps to C_5. Such a homomorphism is given by mapping c_1-vertices to 0, c_2-vertices to 1 and c_3-vertices to 3, and then extending as follows:

- for any vertex u in $N_2(v)$, if u has a c_1-neighbour, map it to 4; otherwise, map it to 2,
- all vertices of $N_1(v)$ are mapped to 3,
- vertex v is mapped to 2 or 4.

Now, let G be a minimal counterexample to Theorem 12. We first collect a few properties of G. The previous paragraph allows us to state our first claim.

Claim 12.A. *For no vertex v of G there is a v-special colouring of $G[N_{3^+}(v)]$.*

Since G is a minimal counterexample, it cannot map to a subgraph of itself (which would be a smaller counterexample):

Claim 12.B. *Graph G is a core. In particular, for any two vertices u and v of G, $N(u) \not\subseteq N(v)$.*

Recall that a *walk* between two vertices u and v is a sequence of (not necessarily distinct) vertices starting with u and ending with v, where two consecutive vertices in the sequence are adjacent. A walk between u and v is an *uv-walk*, and a *k-walk* is a walk with $k+1$ vertices.

Claim 12.C. *For any two distinct vertices u and v of G, there is a uv-walk of length 5.*

Proof of claim. If not, identifying u and v would result in a smaller graph of yet odd-girth 7 which does not map to C_5, contradicting the minimality of G. (□)

Claim 12.D. *Graph G has no thread of length 4 or more.*

Proof of claim. Once again, by minimality of G, if we remove a thread of length 4, the resulting graph maps to C_5. But since there is a walk of length 4 between any two vertices of C_5, this mapping could easily be extended to G. (□)

Now we can state a more difficult claim.

Claim 12.E. *There is no vertex of G of degree 4 or more, nor a vertex of degree exactly 3 with a second neighbourhood of size 5 or more.*

Proof of claim. For a contradiction, suppose that a vertex v has degree 4 or more, or has degree 3 and a second neighbourhood of size 5 or more.

By Claim 12.B, the neighbours of v should have pairwise distinct neighbourhoods, so that even if v has degree 4 or more, we must have $|N_2(v)| \geq 4$. Thus, by a counting argument (recall that G has at most 14 vertices), $N_{3^+}(v)$ has size at most 5. Since G has odd-girth 7, this means $G[N_{3^+}(v)]$ is bipartite.

Suppose $G[N_{3^+}(v)]$ has only one non-trivial connected component. Consider any proper 3-colouring of $G[N_{3^+}(v)]$ such that colours c_1 and c_2 are used for the non-trivial connected component, and colour c_3 is used for isolated vertices. Then, no vertex of $N_2(v)$ can see both colours c_1 and c_2, as this would result in a short odd-cycle in G. Thus, any such colouring is v-special. Such colouring clearly exists, hence by Claim 12.A, we derive that $G[N_{3^+}(v)]$ has at least two non-trivial components. Since it has order at most 5, it must have exactly two.

Assume now that both non-trivial connected components are isomorphic to K_2. Consider all proper 3-colourings of $G[N_{3^+}(v)]$ such that colours c_1 and c_2 are used for the copies of K_2 (the potentially remaining vertex being coloured with c_3). One may check that since by Claim 12.A, none of them is v-special, and hence there is a short odd-cycle in G, which is a contradiction.

Hence, $G[N_{3^+}(v)]$ is isomorphic to the disjoint union of K_2 and $K_{2,1}$. Let u_1 and u_2 be the vertices of K_2, and u_3, u_4 and u_5 be the vertices of $K_{2,1}$ such that u_4 is the central vertex.

Let φ_1 and φ_2 be two proper 2-colourings of $G[N_{3^+}(v)]$ as follows: $\varphi_1(u_1) = \varphi_1(u_3) = \varphi_1(u_5) = c_1$ and $\varphi_1(u_2) = \varphi_1(u_4) = c_2$, $\varphi_2(u_2) = \varphi_2(u_3) = \varphi_2(u_5) = c_1$ and $\varphi_2(u_1) = \varphi_2(u_4) = c_2$. Since φ_1 is not v-special, there is either a vertex t_1 adjacent to u_2 and u_3 (by considering the symmetry of u_3 and u_5), or a vertex t'_1 adjacent to u_1 and u_4. Similarly, since φ_2 is not v-special, either there is a vertex t_2 adjacent to u_1 and one to u_3 and u_5, or there is a vertex t'_2 adjacent to u_2 and u_4. The existence of some t'_i (for $i = 1$ or 2), together with any of these three remaining vertices would result in a short odd-cycle in G. Thus, t_1 and t_2 must exist and more precisely, t_2 has to be a neighbour of u_5.

Next, we show that u_4 has degree at least 3. Suppose not, then it has degree exactly 2. Let φ_3 be a partial C_5-colouring of G defined as follows: $\varphi_3(u_1) = \varphi_3(u_5) = 0, \varphi_3(u_2) = 1, \varphi_3(u_3) = 3$ and $\varphi_3(u_4) = 4$. Then, no vertex in $N_2(v)$ sees both 0 and 1 (by odd-girth arguments). Thus, we can extend φ_3 to $N_2(v)$ using only colours 2 and 4 on these vertices. Then, all vertices of $N_1(v)$ can be mapped to 3 and v can be mapped to 2 and $G \to C_5$, a contradiction.

Hence, there exists a vertex t_3 in $N_2(v)$ which is adjacent to u_4. Note that, by the odd-girth condition, t_3 has no other neighbour in $G[N_{3^+}(v)]$ and, in particular, it must be distinct from t_1 and t_2. Vertices t_1, t_2 and t_3 are in $N_2(v)$, so there are vertices s_1, s_2 and s_3 in $N_1(v)$ such that s_i is adjacent to t_i for i between 1 and 3. Moreover, vertices t_1, t_2 and t_3 are pairwise connected by a path of length 3. Therefore, their neighbourhoods cannot intersect, so that vertices s_1, s_2 and s_3 are distinct.

Now, consider the partial C_5-colouring φ_3 again. We may extend φ_3 to $N_2(v)$ by assigning colour 0 to neighbours of u_4, colour 4 to neighbours of u_1 and u_5, and colour 2 to the rest. If no vertex of $N_1(v)$ sees both colours 0 and 4 in $N_2(v)$, we may colour $N_1(v)$ with 1 and 3 and colour v with 2, which is a contradiction. Thus, there exists some vertex x in $N_1(v)$ seeing both colours 0 and 4 in $N_2(v)$. The only vertices with colour 0 in $N_2(v)$ are neighbours of u_4 so that x must be at distance 2 from u_4. Since there is no short odd-cycle in G, vertex x cannot be at distance 2 from u_5. Thus, it is at distance 2 from u_1. Let t_4 be the middle vertex of this path from x to u_1. Now t_4 is a neighbour of u_1 which is distinct from t_1, t_2 and t_3. By the symmetry between u_1 and u_2, there must be a fifth vertex t_5 in $N_2(v)$ which is a neighbour of u_2 and distinct from t_1, t_2, t_3 and t_4. Moreover, t_5 has a neighbour y in $N_1(v)$ that is at distance 2 from u_4. We can readily check that y is distinct from x, s_1 and s_2. Thus, $N_1(v)$ has size at least 4 and $N_2(v)$ has size at least 5, which is a contradiction with the order of G (which should be at most 14). This concludes the proof of Claim 12.E. (□)

Claim 12.F. *G contains no 6-cycle.*

Proof of claim. Suppose, by contradiction, that G contains a 6-cycle C : $v_0, \ldots, v_5$. For a pair v_i and v_{i+2} (addition in indices are done modulo 6) of vertices of C, the 5-walk connecting them (see Claim 12.C) is necessarily a 5-path, which we denote by P^i. Furthermore, at most one inner-vertex of P^i may belong to C, and if it does, it must be a neighbour of v_i or v_{i+2} (one can check that otherwise, there is a short odd-cycle in G).

Assume first that none of the six paths P^i $(0 \leq i \leq 5)$ has any inner-vertex on C. In this case and by Claim 12.E, we observe that the neighbours of v_i in P^i and P^{i+4} (additions in superscript are modulo 6) are the same. Let v'_i be this neighbour of v_i.

Vertices v'_i , $i = 0, 1, \ldots, 5$ are all distinct, as otherwise we have a short odd-cycle in G. Let x and y by two internal vertices of P^1 distinct from v'_0 and v'_2.

By our assumption, x and y are distinct from vertices of C. We claim that they are also distinct from v'_i, $i = 0, \ldots, 5$. Vertex x is indeed distinct from v'_0 and v'_2 by the choice of P^0. It is distinct from v'_1, v'_3 and v'_4 as otherwise there will be a short odd-cycle. Similarly, y is distinct from $v'_0, v'_1, v'_2, v'_4, v'_5$. For the same reason, we cannot simultaneously have $x = v'_5$ and $y = v'_3$. Finally, if we have $x = v'_5$ then $\{v'_0, v_1, y, v_3, v'_4\} \subseteq N_2(x)$ and $d(x) \geq 3$, contradicting Claim 12.E. As a result, since $|V(G)| \leq 14$, x and y are internal vertices of all P^i's. But then it easy to find a short odd-cycle.

Hence, we may assume, without loss of generality, that P^1 has one inner-vertex on C, say $v'_1 = v_0$. Let $v_0x_1x_2x_3v_3$ be the 4-path connecting v_0 and v_3 (recall that $x_i \notin C$ for $i = 1, 2, 3$). We next assume that P^5 does not have any inner-vertex in C. Then, no vertex of P^5 is a vertex from $\{x_1, x_2, x_3\}$, for otherwise we have a short odd-cycle in G. But then, v_0 violates Claim 12.E. Therefore, an inner-vertex of P^5 lies on C, say it is v_2 and we have $P^5 : v_5y_1y_2y_3v_2v_1$. Then, P^1 and P^5 are vertex-disjoint, for otherwise we have a short odd-cycle in G. Remark that C together with the union of the paths P^i, $i = 0, \ldots, 5$, forms a $(2k+1)$-odd K_4, in fact it is a $(1, 2, 4)$-odd-K_4 (a subgraph of $C(12, 5)$). Now, because of the odd-girth of G, and by Claim 12.E, the only third neighbour of v_1, if any, is v_4 (and vice-versa). Furthermore, the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ of vertices induces a subgraph matching the x_iy_i with $i = 1, 2, 3$. If there is no additional vertex in G, then G has order 12 and it is a subgraph of $C(12, 5)$. But then, the circular chromatic number of G is at most $12/5$, implying that G has a homomorphism to C_5, a contradiction. Thus, G has order at least 13. Again by Claim 12.E, any of the two last potentially existing vertices of G can be adjacent only to x_2 or y_2. Without loss of generality (considering the symmetries of the graph), assume that x_2 has an additional neighbour, v. Then, either v is also adjacent to y_2 (then G has order 13), or v and y_2 have a common neighbour, say w. If v is adjacent to both x_2 and y_2, then there is no edge in the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ (otherwise we have a short odd-cycle). But then, we exhibit a homomorphism of G to C_5: map x_3, y_1, v to 0; x_2, y_2 to 1; v_1, x_1, y_3 to 2; v_0, v_2, v_4 to 3; v_3 and v_5 to 4. This is a contradiction. Thus, v and y_2 have a common neighbour w, and both v and w are of degree 2. We now create a homomorphic image of G by identifying v with y_2 and w with x_2. Then, this image of G is a subgraph of $C(12, 5)$, and thus the circular chromatic number of G is at most $12/5$, implying that G has a homomorphism to C_5, a contradiction. This completes the proof of Claim 12.F. (□)

Claim 12.G. *G contains no* 4*-cycle.*

Proof of claim. Assume by contradiction that G contains a 4-cycle $C : tuvw$. As in the proof of Claim 12.F, there must be two 5-paths $P_{tv} : ta_1a_2a_3a_4v$ and $P_{uw} : ub_1b_2b_3b_4w$ connecting t with v and u with w, respectively. Moreover, these two paths must be vertex-disjoint because of the odd-girth of G. Thus, the union of C, P_{tv} and P_{uw} forms a $(1, 1, 5)$-odd-K_4, K. By assumption on the odd-girth of G, any additional edge inside $V(K)$ must connect an internal vertex of Ptv to an internal vertex of P_{uw}. But any such edge would either create a short odd-cycle or a 6-cycle in G, the latter contradicting Claim 12.F. Thus, the only edges in $V(K)$ are those of K. If there is no additional vertex in G, we have two 5-threads in G, contradicting Claim 12.D; thus there is at least one additional vertex in G, say x, and perhaps a last vertex, y. Note that t, u, v and w are already, in K, degree 3-vertices with a second neighbourhood of size 4, thus by Claim 12.E none of t, u, v, w, a_1, a_4, b_1 and b_4 are adjacent to any vertex not in K. Thus $N(x), N(y) \subseteq \{a_2, a_3, b_2, b_3, x, y\}$.

Assume that some vertex not in K (say x) is adjacent to two vertices of K. Then, these two vertices must be a vertex on the tu-path of K (a_2 or a_3, without loss of generality it is a_2) and a vertex on the vw-path of K (b_2 or b_3). By the automorphism of K that swaps v and w and reverses the vw-path, without loss of generality we can assume that the second neighbour of x in K is b_3. Then, we claim that G has no further vertex. Indeed, if there is a last vertex y, since a_2 and b_3 have already three neighbours, y must be adjacent to at least two vertices among $\{a_3, b_2, x\}$. If it is adjacent to both a_3 and b_2, we have a 6-cycle, contradicting Claim 12.F; otherwise, y is of degree 2 but part of a 4-cycle, implying that G is not a core, a contradiction. Thus, G has order 13 and no further edge. We create a homomorphic image of G by identifying x and a_3. The obtained graph is a subgraph of $C(12, 5)$. Hence, G has circular chromatic number at most 12/5 and a homomorphism to C_5, a contradiction.

Thus, any vertex not in K has at most one neighbour in K. Since G has no 4-thread, one vertex not in K (say x) is adjacent to one of a_2 or a_3 (without loss of generality, say a_2), and the last vertex, y, is adjacent to one of b_2 and b_3 (as before, by the symmetries of K we can assume it is b_2). Moreover, x and y must be adjacent, otherwise they both have degree 1. Also there is no further edge in G. But then, as before, we create a homomorphic image of G by identifying x with b_2 and y with a_2. The resulting graph is a subgraph of $C(12, 5)$, which again gives a contradiction. This completes the proof of Claim 12.G. (□)

To complete the proof, we note that since G has no homomorphism to C_5, it also has no homomorphism to C_7. Thus, by Theorem 5, G must contain either an odd-K_3^2 or an odd-K_4 of odd-girth at least 7. Since such an odd-K_3^2 must have at least 18 vertices, G contains an odd-K_4. Let H be such an odd-K_4 of G. Since its girth is at least 7, by Proposition 4 it has at least 12 vertices. We consider three cases, depending on the order of H. Due to the space limit, the proofs of these three cases are omitted but can be found in the full version of the paper. □

We now deduce the following consequence of Theorem 12 and Proposition 9, that improves the known lower bounds on $\eta(4, 2)$ and $\eta(4, 1)$ (noting that for larger values of k, the bound of Corollary 8 is already stronger).

Corollary 13. *We have* $\eta(4, 1) \geq \eta(4, 2) \geq 17$.

5 Concluding Remarks

In this work, we have started investigating the smallest order of a C_{2l+1}-critical graph of odd-girth $2k+1$. We have determined a number of previously unknown values, in particular we showed that a smallest C_5-critical graph of odd-girth 7 is of order 15. In contrast to the result of Chvátal on the uniqueness of the smallest triangle-free 4-chromatic graph [3], we have found more than one such graph: Gordon Royle showed computationally, that there are at least 10 such graphs (private communication, 2016).

Regarding Table 1, we do not know the growth rate in each row of the table. Perhaps it is quadratic; that would be to say that for a fixed l, $\eta(k,l) = \Theta(k^2)$. This is indeed true for $l = 1$, as proved by Nilli [17].

Our last remark is about Theorem 10. We think that for any given k, Theorem 10 covers all values of k for which $\eta(k,l) = 4k$.

References

1. Beaudou, L., Foucaud, F., Naserasr, R.: Homomorphism bounds and edge-colourings of K_4-minor-free graphs. J. Comb. Theor. Ser. B **124**, 128–164 (2017)
2. Catlin, P.A.: Graph homomorphisms into the five-cycle. J. Comb. Theor. Ser. B **45**, 199–211 (1988)
3. Chvátal, V.: The minimality of the mycielski graph. In: Bari, R.A., Harary, F. (eds.) Graphs and Combinatorics. LNM, vol. 406, pp. 243–246. Springer, Heidelberg (1974). https://doi.org/10.1007/BFb0066446
4. Dirac, G.A.: A property of 4-chromatic graphs and remarks on critical graphs. J. London Math. Soc. **27**, 85–92 (1952)
5. Erdős, P.: Graph theory and probability. Can. J. Math. **11**, 34–38 (1959)
6. Exoo, G., Goedgebeur, J.: Bounds for the smallest k-chromatic graphs of given girth. Discrete Math. Theor. Comput. Sci. **21**(3), 9 (2019)
7. Gallai, T.: Kritische Graphen I. Magyar Tud. Akad. Mat. Kutató Int. Közl. **8**, 165–192 (1963)
8. Gallai, T.: Kritische Graphen II. Magyar Tud. Akad. Mat. Kutató Int. Közl. **8**, 373–395 (1963)
9. Gerards, A.M.H.: Homomorphisms of graphs into odd cycles. J. Graph Theor. **12**(1), 73–83 (1988)
10. Gyárfás, A., Jensen, T., Stiebitz, M.: On graphs with strongly independent colour classes. J. Graph Theor. **46**(1), 1–14 (2004)
11. Harary, F.: Graph Theory, p. 149. Addison-Wesley, Reading (1969). Exercise 12.19
12. Hell, P., Nešetřil, J.: Graphs and Homomorphisms. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, Oxford (2004)
13. Jiang, T.: Small odd cycles in 4-chromatic graphs. J. Graph Theor. **37**(2), 115–117 (2001)
14. Mycielski, J.: Sur le coloriage des graphes. Colloq. Math. **3**, 161–162 (1955)
15. Năstase, E., Rödl, V., Siggers, M.: Note on robust critical graphs with large odd girth. Discrete Math. **310**(3), 499–504 (2010)
16. Ngoc, N.V., Tuza, Z.: 4-chromatic graphs with large odd girth. Discrete Math. **138**(1–3), 387–392 (1995)
17. Nilli, A.: Short odd cycles in 4-chromatic graphs. J. Graph Theor. **31**(2), 145–147 (1999)
18. Payan, C.: On the chromatic number of cube-like graphs. Discrete Math. **103**(3), 271–277 (1992)
19. Tardif, C.: The fractional chromatic numbers of cones over graphs. J. Graph Theor. **38**(2), 87–94 (2001)

20. Youngs, D.A.: 4-chromatic projective graphs. J. Graph Theor. **21**(2), 219–227 (1996)
21. Zhu, X.: Circular chromatic number, a survey. Discrete Math. **229**(1–3), 371–410 (2001)