Location-domination and metric dimension in interval and permutation graphs

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joint work with:

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April 2015
Location-domination
Fire detection in a building

Detecto r can detect re in its ro om and its neighb o rho o d (through a do o r). Each ro om must contain a detecto r o r have one in an adjacent ro om.
Detector can detect fire in its room and its neighborhood (through a door).
Fire detection in a building

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Each room must contain a detector or have one in an adjacent room.
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Modelisation with a graph

Graph $G = (V, E)$. Vertices: rooms.
Edges: between any two rooms connected by a door
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- Set of detectors = dominating set $D \subseteq V$: $\forall u \in V, N[u] \cap D \neq \emptyset$
Modelisation with a graph

Graph $G = (V, E)$. Vertices: rooms. Edges: between any two rooms connected by a door.

Set of detectors = dominating set $D \subseteq V$: $\forall u \in V, N[u] \cap D \neq \emptyset$

Domination number $\gamma(G)$: smallest size of a dominating set of $G$
Back to the building

Where is the fire?
Where is the fire?
Back to the building

Where is the fire?
Where is the fire?

To locate the fire, we need more detectors.
Locating the fire
In each room with no detector, set of dominating detectors is distinct.
Peter Slater, 1980’s. **Locating-dominating set** $D$:
subset of vertices of $G = (V, E)$ which is:

- dominating: $\forall u \in V, N[u] \cap D \neq \emptyset$,
- locating: $\forall u, v \in V \setminus D, N[u] \cap D \neq N[v] \cap D$. 

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Identification problems in graphs

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$\gamma_L(G)$: location-domination number of $G$, minimum size of a locating-dominating set of $G$. 
Locating the fire

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**$\gamma_L(G)$**: location-domination number of $G$, minimum size of a locating-dominating set of $G$.

**Remark**: $\gamma(G) \leq \gamma_L(G)$
Examples: paths

Domination number: $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Location-domination number: $\gamma_L(P_n) = \left\lceil \frac{2n}{5} \right\rceil$
Theorem (Slater, 1980’s)

If $G$ is a graph of order $n$, $\gamma_L(G) = k$. Then $n \leq 2^k + k - 1$, i.e. $\gamma_L(G) = \Omega(\log n)$. 
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Tight example ($k = 4$):
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**Theorem (Slater, 1980’s)**

$G$ tree of order $n$, $\gamma_L(G) = k$. Then $n \leq 3k - 1$, i.e. $\gamma_L(G) \geq \frac{n+1}{3}$.

**Theorem (Rall & Slater, 1980’s)**

$G$ planar graph, order $n$, $\gamma_L(G) = k$. Then $n \leq 7k - 10$, i.e. $\gamma_L(G) \geq \frac{n+10}{7}$.

Tight examples:
**Definition - Interval graph**

Intersection graph of intervals of the real line.
Lower bound for interval graphs

**Theorem** (F., Mertzios, Naserasr, Parreau, Valicov)

Let $G$ be an interval graph of order $n$, $\gamma_L(G) = k$.

Then $n \leq \frac{k(k+3)}{2}$, i.e. $\gamma_L(G) = \Omega(\sqrt{n})$. 

Lower bound for interval graphs

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- Locating-dominating $D$ of size $k$.
- Define zones using the right points of intervals in $D$. 
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- Locating-dominating $D$ of size $k$.
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- Each vertex intersects a consecutive set of intervals of $D$ when ordered by left points.
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A lower bound for interval graphs

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- Locating-dominating set $D$ of size $k$.
- Define zones using the right points of intervals in $D$.
- Each vertex intersects a consecutive set of intervals of $D$ when ordered by left points.

$$n \leq \sum_{i=1}^{k} (k-i) + k = \frac{k(k+3)}{2}.$$
Lower bound for interval graphs

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Tight:
Definition - Permutation graph

Given two parallel lines $A$ and $B$: intersection graph of segments joining $A$ and $B$. 

![Diagram of permutation graph](image)
Lower bound for permutation graphs

**Theorem (F., Mertzios, Naserasr, Parreau, Valicov)**

Let $G$ be a permutation graph of order $n$, $\gamma_L(G) = k$.

Then $n \leq k^2 + k - 2$, i.e. $\gamma_L(G) = \Omega(\sqrt{n})$. 

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Identification problems in graphs

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**Theorem** (F., Mertzios, Naserasr, Parreau, Valicov)

A permutation graph of order \( n \), \( \gamma_L(G) = k \).

Then \( n \leq k^2 + k - 2 \), i.e. \( \gamma_L(G) = \Omega(\sqrt{n}) \).

- Locating-sominating set \( D \) of size \( k \): \( k + 1 \) "top zones" and \( k + 1 \) "bottom zones"
- Only one segment in \( V \setminus D \) for one pair of zones
  \[ \rightarrow n \leq (k+1)^2 + k \]
- Careful counting for the precise bound
Lower bound for permutation graphs

**Theorem (F., Mertzios, Naserasr, Parreau, Valicov)**

Let \( G \) be a permutation graph of order \( n \), with \( \gamma_L(G) = k \). Then \( n \leq k^2 + k - 2 \), i.e. \( \gamma_L(G) = \Omega(\sqrt{n}) \).

Tight:
Metric dimension
Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them
Determination of Position in 3D Euclidean space

**GPS/GLONASS/Galileo/Beidou/IRNSS:**

need to know the exact position of 4 satellites + distance to them

**Question**

Does the “GPS” approach also work in undirected unweighted graphs?
Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $\text{dist}(w, u) \neq \text{dist}(w, v)$

**Definition** - Resolving set (Slater, 1975 - Harary & Melter, 1976)

$R \subseteq V(G)$ resolving set of $G$:

$\forall u \neq v$ in $V(G)$, there exists $w \in R$ that distinguishes $\{u, v\}$. 
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![Graph with nodes and edges to illustrate resolving set](image)
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**MD**(\( G \)): metric dimension of \( G \), minimum size of a resolving set of \( G \).
Any locating-dominating set is a resolving set, hence $MD(G) \leq \gamma_L(G)$.

A locating-dominating set can be seen as a “distance-1-resolving set”.
Remarks

• Any locating-dominating set is a resolving set, hence $MD(G) \leq \gamma_L(G)$.

• A locating-dominating set can be seen as a “distance-1-resolving set”.

**Proposition**

$MD(G) = 1 \iff G$ is a path
Example of path: no bound $n \leq f(MD(G))$ possible.
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**Theorem (Khuller, Raghavachari & Rosenfeld, 2002)**

$G$ of order $n$, diameter $D$, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter: maximum distance between two vertices)
Example of path: no bound \( n \leq f(MD(G)) \) possible.

**Theorem (Khuller, Raghavachari & Rosenfeld, 2002)**

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**Theorem (F., Mertzios, Naserasr, Parreau, Valicov)**

\[ G \text{ interval graph or permutation graph of order } n, MD(G) = k, \text{ diameter } D. \text{ Then } n = O(Dk^2) \text{ i.e. } k = \Omega\left(\sqrt{\frac{n}{D}}\right). \]

→ Proofs are similar as for locating-dominating sets.
→ Bounds are tight (up to constant factors).
Algorithmic complexity
LOCATING-DOMINATING SET

**INPUT:** Graph $G$, integer $k$.

**QUESTION:** Is there a locating-dominating set of $G$ of size $k$?
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**Theorem (F., Mertzios, Naserasr, Parreau, Valicov)**

LOCATING-DOMINATING SET is NP-complete for graphs that are both interval and permutation.

Reduction from 3-DIMENSIONAL MATCHING.

**Main idea:** an interval can separate pairs of intervals far away from each other (without affecting what lies in between)
Interval and permutation graphs

**METRIC DIMENSION**

**INPUT:** Graph $G$, integer $k$.

**QUESTION:** Is there a resolving set of $G$ of size $k$?

**Reduction from LOCATING-DOMINATING SET to METRIC DIMENSION:**

$$\text{MD}(G') = \gamma_{L}(G) + 2$$

**Corollary** (F., Mertzios, Naserasr, Parreau, Valicov)

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Identification problems in graphs
**Interval and permutation graphs**

**Metric Dimension**

**INPUT:** Graph $G$, integer $k$.

**QUESTION:** Is there a resolving set of $G$ of size $k$?

Reduction from **LOCATING-DOMINATING SET** to **METRIC DIMENSION**:

$$MD(G') = \gamma_L(G) + 2$$

**Corollary** (F., Mertzios, Naserasr, Parreau, Valicov)

**METRIC DIMENSION** is NP-complete for graphs that are both interval and permutation (and have diameter 2).
Note: METRIC DIMENSION W[2]-hard even for subcubic bipartite graphs
\[\rightarrow\] probably no \( f(k)\text{poly}(n) \)-time algorithm

**Theorem (F., Mertzios, Naserasr, Parreau, Valicov)**

METRIC DIMENSION can be solved in time \( 2^{O(k^4)}n \) on interval graphs.

Ideas:
- use dynamic programming on a path-decomposition of \( G^4 \).
- each bag has size \( O(k^2) \).
- it suffices to separate vertices at distance 2
- “transmission” lemma for separation constraints
Open problems

- Investigate bounds for other “geometric” graphs, for $MD$ and $\gamma_L$

- Complexity of LOCATING-DOMINATING SET, METRIC DIMENSION on unit interval graphs

- Complexity of METRIC DIMENSION for bounded treewidth

- Parameterized complexity of METRIC DIMENSION: planar graphs, chordal graphs, permutation graphs...
Open problems

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THANKS FOR YOUR ATTENTION
Complexity of LOCATING-DOMINATING SET

- NP-complete
- polynomial
- trees, cographs
- bounded treewidth, bounded cliquewidth
- perfect, chordal, co-comparability, split, interval
- permutation, co-bipartite, line of bipartite
- claw-free, quasi-line, line, unit interval
- planar bipartite

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Identification problems in graphs
Complexity of METRIC DIMENSION

- **NP-complete**
  - planar
  - series-parallel
  - outerplanar

- **polynomial**
  - trees
  - bounded cyclomatic number
  - bounded vertex cover
  - bounded distance to linear forest
  - bounded distance to forest (FVS)
  - bounded pathwidth
  - bounded treewidth
  - bounded cliquewidth

- **OPEN**
  - perfect
  - chordal
  - co-comparability
  - split
  - co-bipartite
  - line of bipartite
  - line
  - quasi-line

- **unit interval**
  - co-bipartite
  - permutation
  - interval

- **claw-free**
  - co-compatability
  - compatability
  - chordal
  - claw-free
  - perfect
  - permutation

- **NP-complete**
  - identification problems in graphs