Edge identifying codes
(identified codes in line graphs)

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Locating a burglar in a math department
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How many detectors do we need?
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Edge identifying codes
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Edge identifying codes
How many detectors do we need?
Identifying codes: definition

Let \( N[u] \) be the set of vertices \( v \) s.t. \( d(u, v) \leq 1 \)

**Definition** - Identifying code of \( G \) (Karpovsky, Chakrabarty, Levitin, 1998)

Subset \( C \) of \( V \) such that:
- \( C \) is a dominating set in \( G \): \( \forall u \in V, N[u] \cap C \neq \emptyset \), and
- \( C \) is a separating code in \( G \): \( \forall u \neq v \) of \( V \), \( N[u] \cap C \neq N[v] \cap C \)
Let $N[u]$ be the set of vertices $v$ s.t. $d(u, v) \leq 1$

**Definition** - Identifying code of $G$ (Karpovsky, Chakrabarty, Levitin, 1998)

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**Notation** - Identifying code number

$\gamma^{ID}(G)$: minimum cardinality of an identifying code of $G$
Let $N[u]$ be the set of vertices $v$ s.t. $d(u, v) \leq 1$

**Remark**

Not all graphs have an identifying code!

Twins = pair $u, v$ such that $N[u] = N[v]$.

A graph is identifiable iff it is twin-free (i.e. it has no twins).
Identifiable graphs

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**Twins** = pair $u$, $v$ such that $N[u] = N[v]$.

A graph is **identifiable** iff it is **twin-free** (i.e. it has no twins).
Let $G$ be an identifiable graph, then

$$\lceil \log_2(n + 1) \rceil \leq \gamma^{\text{ID}}(G)$$

**Theorem** (Karpovsky, Chakrabarty, Levitin, 1998)
Bounds

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**Theorem (Gravier, Moncel, 2007)**

Let $G$ be an identifiable graph with at least one edge, then

$$\gamma^{ID}(G) \leq n - 1$$
Bounds

**Theorem (Karpovsky, Chakrabarty, Levitin, 1998)**

Let $G$ be an identifiable graph, then

$$\lceil \log_2(n + 1) \rceil \leq \gamma_{ID}(G)$$

**Theorem (Gravier, Moncel, 2007)**

Let $G$ be an identifiable graph with at least one edge, then

$$\gamma_{ID}(G) \leq n - 1$$

Both bounds are tight, and all extremal examples are known:

- lower bound: Moncel, 2006
Let $I[e]$ be the set of edges $f$ s.t. $e = f$ or $e, f$ are incident to a common vertex

**Definition - Edge identifying code of $G$ (without isolated vertices)**

Subset $C_E$ of $E$ such that:
- $C_E$ is an edge dominating set in $G$: $\forall e \in E$, $I[e] \cap C_E \neq \emptyset$, and
- $C_E$ is an edge separating code in $G$: $\forall e \neq f$ of $E$, $I[e] \cap C_E \neq I[f] \cap C_E$
Let $I[e]$ be the set of edges $f$ s.t. $e = f$ or $e, f$ are incident to a common vertex

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**Remark**

Edge identifying code of $G \longleftrightarrow$ Identifying code of $\mathcal{L}(G)$
Let $I[e]$ be the set of edges $f$ s.t. $e = f$ or $e, f$ are incident to a common vertex.

**Definition - Edge identifying code of $G$ (without isolated vertices)**

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**Remark**

Edge identifying code of $G \iff$ Identifying code of $\mathcal{L}(G)$

**Notation - Edge identifying code number**

$\gamma^{ID}(\mathcal{L}(G)) = \gamma^{EID}(G)$: minimum cardinality of an edge identifying code of $G$
Not all graphs have an edge identifying code!

**Pendant** = pair of twin edges.

A graph is **edge identifiable** iff it is **pendant-free** (and simple).
Lower bounds

Theorem (F., Gravier, Naserasr, Parreau, Valicov)

Let $G$ be an edge identifiable graph with an edge identifying code $C_E$ inducing a connected subgraph, then $|E(G)| \leq \left(\frac{|C_E|+2}{2}\right)^2 - 4$
Lower bounds

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**Theorem (F., Gravier, Naserasr, Parreau, Valicov)**

Let $G$ be an edge identifiable graph with an edge identifying code of size $k$, then $|E(G)| \leq \begin{cases} \frac{4}{3}k, & \text{if } k \equiv 0 \mod 3 \\ \frac{4}{3}(k-1)+1, & \text{if } k \equiv 1 \mod 3 \\ \frac{4}{3}(k-2)+2, & \text{if } k \equiv 2 \mod 3 \end{cases}$
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**Corollary**

$\gamma^{ID}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4} \sqrt{|V(\mathcal{L}(G))|}$. This bound is tight.
Let $G$ be an edge identifiable graph with an edge identifying code $C_E$ inducing a connected subgraph, then $|E(G)| \leq \left(\frac{|C_E|+2}{2}\right) - 4$

Theorem (F., Gravier, Naserasr, Parreau, Valicov)

Let $G' = G[C_E]$. Each edge $uv \in G$ is determined by two sets:
- set of edges of $G'$ incident to $u$
- set of edges of $G'$ incident to $v$

At most $|V(G')| + \left(\frac{|V(G')|}{2}\right) = \left(\frac{|V(G')|+1}{2}\right)$ such sets.

- $G'$ not a tree $\Rightarrow |V(G')| \leq |C_E|$
- $G'$ tree: we show that at least 4 of these sets cannot be used.
Corollary

\[ \gamma^{\text{ID}}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4} \sqrt{|V(\mathcal{L}(G))|}. \] This bound is tight.
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\[ \gamma^{ID}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4} \sqrt{|V(\mathcal{L}(G))|}. \] This bound is tight.

**Theorem** *(Beineke, 1970)*

\( G \) is a line graph if and only if it does not contain one of the following graphs as an induced subgraph.

[Diagram of graphs showing examples of line graphs and forbidden subgraphs]
**Corollary**

\[ \gamma^{ID}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4} \sqrt{|V(\mathcal{L}(G))|} \]. This bound is tight.

**Theorem (Beineke, 1970)**

$G$ is a line graph if and only if it does not contain one of the following graphs as an induced subgraph.

The bound does **not** hold for claw-free graphs.

**Question**

Does the bound hold for a class defined by a smaller subfamily of the list?
An upper bound

Theorem (F., Gravier, Naserasr, Parreau, Valicov)

Let $G$ be an edge-identifiable graph with a minimal edge identifying code $C_E$. Then $G[C_E]$ is 2-degenerated.
An upper bound

**Theorem** (F., Gravier, Naserasr, Parreau, Valicov)

Let $G$ be an edge-identifiable graph with a minimal edge identifying code $C_E$. Then $G[C_E]$ is 2-degenerated.

**Corollary**

If $G$ is an edge-identifiable graph on $n$ vertices not isomorphic to $K_4^-$, then $\gamma^{EID}(G) \leq 2|V(G)| - 4$. 

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An upper bound

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If $G$ is an edge-identifiable graph on $n$ vertices not isomorphic to $K_4^-$, then $\gamma^{EID}(G) \leq 2|V(G)| - 4$.

This is almost tight since $\gamma^{EID}(K_{2,n}) = 2n - 2 = 2|V(K_{2,n})| - 6$. 

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An upper bound - corollary

**Corollary**

If $G$ is an edge-identifiable graph on $n$ vertices not isomorphic to $K_4^-$, then $\gamma_{EID}(G) \leq 2|V(G)| - 4$.

**Corollary**

If $G$ is an edge-identifiable graph with average degree $\overline{d}(G) \geq 5$, then $\gamma_{ID}(\mathcal{L}(G)) \leq n - n \frac{n}{\Delta(\mathcal{L}(G))}$ where $n = |V(\mathcal{L}(G))|$. 

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

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Edge identifying codes
**An upper bound - corollary**

**Corollary**

If $G$ is an edge-identifiable graph on $n$ vertices not isomorphic to $K_4^-$, then $\gamma^{\text{EID}}(G) \leq 2|V(G)| - 4$.

**Corollary**

If $G$ is an edge-identifiable graph with average degree $\bar{d}(G) \geq 5$, then $\gamma^{\text{ID}}(L(G)) \leq n - \frac{n}{\Delta(L(G))}$ where $n = |V(L(G))|$.

**Conjecture** (F., Klasing, Kosowski, Raspaud, 2009)

Let $G$ be a connected identifiable graph on $n$ vertices and of maximum degree $\Delta$. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + O(1)$. 
Problem **EDGE IDCODE**

**INSTANCE:** A graph $G$ and an integer $k$.

**QUESTION:** Does $G$ have an edge identifying code of size at most $k$?

**Theorem** (F., Gravier, Naserasr, Parreau, Valicov)

EDGE IDCODE is NP-complete, even for planar subcubic bipartite graphs of arbitrarily large girth.
Problem EDGE IDCODE

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EDGE IDCODE is NP-complete, even for planar subcubic bipartite graphs of arbitrarily large girth.
Proof by reduction from:

**Problem PLANAR \((\leq 3, 3)\)-SAT**

**INSTANCE:** A set \(Q\) of clauses over a set \(X\) of boolean variables such that:

- Each clause contains at least two and at most three distinct literals
- Each variable appears exactly once negated, twice non-negated
- The bipartite incidence graph \(B(Q)\) is planar

**QUESTION:** Can \(Q\) be satisfied, i.e. is there a truth assignment of the variables of \(X\) such that each clause contains at least one true literal?

**Theorem** (Dahlhaus, Johnson, Papadimitriou, Seymour, Yannakakis, 1994)

PLANAR \((\leq 3, 3)\)-SAT is NP-complete.
Reduction

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$Q$ is satisfiable if and only if $G$ contains an edge identifying code $C_E$ of size $k = 25|Q| + 22|X|$. 
A line graph $\mathcal{L}(G)$ is perfect if and only if $G$ has no odd cycles of length more than 3.
Complexity

**Theorem (Trotter, 1977)**
A line graph $\mathcal{L}(G)$ is perfect if and only if $G$ has no odd cycles of length more than 3.

**Corollary**
IDCODE is NP-complete even when restricted to perfect 3-colorable planar line graphs of maximum degree 4.
Complexity

**Theorem (Courcelle, 1990)**

Every graph property expressable in monadic second-order logic is solvable in linear time in classes of graphs having bounded tree-width.

**Corollary**

EDGE IDCODE is linear time solvable in trees, $k$-outerplanar graphs, series-parallel graphs, ...

**Graph**

- set $V$ of vertices, set $E$ of edges, unary predicates $a, b : E \rightarrow V$
- $e \neq f := (a(e) \neq a(f) \land a(e) \neq b(f)) \lor (b(e) \neq a(f) \land b(e) \neq b(f))$
- $e \mathcal{I}^* f := a(e) = a(f) \lor a(e) = b(f) \lor b(e) = b(f) \lor b(e) = a(f)$

$$\exists C, C \subseteq E, |C| \leq k, \left( \forall e \in E, \exists f \in C \land e \mathcal{I}^* f \right) \land$$

$$\left( \forall e \in E, \forall f \in E, e \neq f, \exists g \in C, ((e \mathcal{I}^* g \land \neg (f \mathcal{I}^* g)) \lor (f \mathcal{I}^* g \land \neg (e \mathcal{I}^* g))) \right)$$
Gràcies!