Complexity of the identifying code problem in restricted graph classes

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IWOCA
The test cover problem

**Definition - TEST COVER** (mentioned in Garey, Johnson, 1979)

**INPUT:** set system (i.e. hypergraph) \((X, S)\)

**TASK:** find the minimum subset \(T \subseteq S\) such that each element \(x \in X\) belongs to a different set of sets in \(T\).

\(X\) (elements)
- \((\emptyset)\) \(a\)
- \({\{2, 3\}}\) \(b\)
- \({\{3\}}\) \(c\)
- \({\{3, 5\}}\) \(d\)

\(S\) (tests)
- \(1 = \{a, b\}\)
- \(2 = \{b\}\)
- \(3 = \{b, c, d\}\)
- \(4 = \{c, d\}\)
- \(5 = \{d\}\)

Example: \(T = \{2, 3, 5\}\)
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**Remark**

Equivalently: for any pair \(x, y\) of elements of \(X\), there is a set in \(T\) that contains **exactly** one of \(x, y\), i.e. the symmetric difference of the sets of tests covering \(x, y\) is **nonempty**.
Given a set system $(X, S)$, a solution to TEST COVER has size at least $\log_2(|X|)$.

**Theorem (Folklore)**

**Proof:** Must assign to each element of $X$, a distinct subset of $T$. Hence $|X| \leq 2^{|T|}$. 


General bounds

**Theorem (Folklore)**

Given a set system \((X, S)\), a solution to TEST COVER has size at least \(\log_2(|X|)\).

**Proof:** Must assign to each element of \(X\), a distinct subset of \(T\). Hence \(|X| \leq 2^{|T|}\).

**Theorem (Bondy’s theorem, 1972)**

Given a set system \((X, S)\), a minimal solution to TEST COVER has size at most \(|X| - 1\).

**Proof:** nice and short graph-theoretic argument.
Complexity results

**Theorem (Garey, Johnson, 1979)**

TEST COVER is NP-complete.

**Theorem (Charon, Cohen, Hudry, Lobstein, 2008)**

TEST COVER is NP-complete, even for set systems with a planar incidence graph.
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**Theorem** (De Bontridder, Haldorsson, Haldorsson, Hurkens, Lenstra, Ravi, Stougie, 2003)

MIN TEST COVER is \(O(\log(|X|))\)-approximable, but NP-hard to approximate within \(o(\log(|X|))\).

**Proof:** Reductions from and to MIN SET COVER.
A special case: identifying the rooms of a building

\[ \text{Graph } G = (V, E) \]

\( V \): vertices (rooms),
\( E \subseteq V \times V \): edges (doors)
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Identifying codes, a special case of test covers

\( G \): undirected graph  
\( N[u] \): set of vertices \( v \) s.t. \( d(u, v) \leq 1 \)

**Definition** - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset \( C \) of \( V(G) \) such that:
- \( C \) is a **dominating set** in \( G \): \( \forall u \in V(G), N[u] \cap C \neq \emptyset \), and
- \( C \) is a **separating code** in \( G \): \( \forall u \neq v \) of \( V(G), N[u] \cap C \neq N[v] \cap C \)
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Computational problems

**Definition - MIN IDCODE**

**INPUT:** graph $G$

**TASK:** find a minimum-size identifying code of $G$

**Theorem (Cohen, Honkala, Lobstein, Zémor, 1999)**

MIN IDCODE NP-hard (reduction from 3SAT).

NP-completeness also holds for planar subcubic graphs, planar bipartite unit disk graphs, line graphs, etc.
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**Theorem** (Berger-Wolf, Laifenfeld, Trachtenberg, 2006)

MIN IDCODE is approximable within $O(\log(n))$, but NP-hard to approximate within $o(\log(n))$ (reduction from MIN SET COVER).
A simple reduction from MIN VERTEX COVER

**Reduction:** subdivide each edge $xy$ of $G$ once, add pendant vertex.
A simple reduction from MIN VERTEX COVER

**Reduction:** subdivide each edge $xy$ of $G$ once, add pendant vertex.

![Diagram of a graph with edges $x$ and $y$ and a pendant vertex](image)

**Proposition**

If $G$ has min. degree 2, $G$ has a vertex cover of size $k$ iff $f(G)$ has an id. code of size $k + |E(G)|$. 
A simple reduction from MIN VERTEX COVER

**Reduction:** subdivide each edge $xy$ of $G$ once, add pendant vertex.

If $G$ has min. degree 2, $G$ has a vertex cover of size $k$ iff $f(G)$ has an id. code of size $k + |E(G)|$.

**Proposition**

MIN VERTEX COVER hard for planar cubic graphs.

**Theorem (F.)**

MIN IDCODE is NP-hard for subcubic bipartite planar graphs.
New non-approximability reductions

Reduction: MIN TEST COVER to MIN IDCODE for bipartite graphs.

If MIN IDCODE has an $\alpha$-approximation algorithm, then MIN TEST COVER has a $4\alpha$-approximation algorithm.

Theorem (F.)

Proof:

Build approximate id. code $C$ with $|C| \leq \alpha \cdot \text{OPT ID}$

Build test cover $T$: $|T| \leq \alpha \cdot \text{OPT ID} - 3 \log_2(|S|) - 2 \leq \alpha \cdot (\text{OPT TC} + 3 \log_2(|S|) + 2) - 3 \log_2(|S|) - 2 \leq \alpha \cdot \text{OPT TC} + (\alpha - 1)3 \log_2(|S|) \leq 4\alpha \cdot \text{OPT TC}$

It is NP-hard to approximate MIN IDCODE within $o(\log(n))$, even for bipartite graphs.
New non-approximability reductions

Reduction: MIN TEST COVER to MIN IDCODE for bipartite graphs.

$$(X, S)$$ has a test cover of size $k$ if and only if $G(X, S)$ has an identifying code of size $k + 3\lceil\log_2(|S| + 1)\rceil + 2$. Constructive.

If MIN IDCODE has an $\alpha$-approximation algorithm, then MIN TEST COVER has a $4\alpha$-approximation algorithm.
New non-approximability reductions

Reduction: MIN TEST COVER to MIN IDCODE for bipartite graphs.

\[(X, S)\] has a test cover of size \(k\) if and only if \(G(X, S)\) has an identifying code of size \(k + 3\lceil \log_2(|S| + 1) \rceil + 2\). Constructive.

If MIN IDCODE has an \(\alpha\)-approximation algorithm, then MIN TEST COVER has a \(4\alpha\)-approximation algorithm.

**Proof:** Build approximate id. code \(C\) with \(|C| \leq \alpha \text{OPT}_{ID}\)

Build test cover \(T\): \(|T| \leq \alpha \text{OPT}_{ID} - 3 \log_2(|S|) - 2\)

\[\leq \alpha (\text{OPT}_{TC} + 3 \log_2(|S|) + 2) - 3 \log_2(|S|) - 2\]

\[\leq \alpha \text{OPT}_{TC} + (\alpha - 1)3 \log_2(|S|)\]

\[\leq 4\alpha \text{OPT}_{TC}\]
New non-approximability reductions

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\[ (X, S) \text{ has a test cover of size } k \text{ if and only if } G(X, S) \text{ has an identifying code of size } k + 3 \lceil \log_2(|S| + 1) \rceil + 2. \text{ Constructive.} \]

\[ \text{If MIN IDCODE has an } \alpha\text{-approximation algorithm, then MIN TEST COVER has a } 4\alpha\text{-approximation algorithm.} \]

Corollary

It is NP-hard to approximate MIN IDCODE within \( o(\log(n)) \), even for bipartite graphs.
New non-approximability reductions

Similar reductions for split graphs and co-bipartite graphs.

**Theorem (F.)**

It is NP-hard to approximate MIN IDCODE within $o(\log(n))$, even for split graphs and even for co-bipartite graphs.
Interval graphs

**Definition - Interval graph**

Intersection graph of intervals of the real line.
Interval graphs

**Definition - Interval graph**

Intersect of intervals of the real line.

**Theorem (F., Mertzios, Valicov)**

MIN IDCODE is NP-hard for interval graphs. Reduction from 3-DIMENSIONAL MATCHING.
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**Theorem** (F., Mertzios, Valicov)

MIN IDCODE is NP-hard for interval graphs. Reduction from 3-DIMENSIONAL MATCHING.

**Main idea**: an interval can separate pairs of intervals lying far away from each other (without affecting what lies in between).
MIN IDCODE for unit interval graphs

**Definition - Unit interval graph**

Intersection graph of intervals of the real line all having unit length. Equivalent to *proper* interval graphs (no interval contains another).
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**Observation**

Our reduction creates interval graphs that are far from proper/unit.
MIN IDCODE for unit interval graphs

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Intersection graph of intervals of the real line all having unit length. Equivalent to *proper* interval graphs (no interval contains another).

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Our reduction creates interval graphs that are far from proper/unit.

**Question**

What is the complexity of MIN IDCODE for *unit* interval graphs?
MIN IDCODE for unit interval graphs

**Definition - Ladder graph** $L_m$

$L_m$ is the grid graph $P_2 \square P_m$.

**Definition - Cycle cover**

Set $S$ of cycles of graph $G$ s.t. $\bigcup_{S \in S} E(S) = E(G)$.
**MIN IDCODE for unit interval graphs**

**Definition - LADDER CYCLE COVER**

**INPUT:** An integer $m$ and an integer $k$, and a set $S$ of cycles of $L_m$.

**TASK:** Find a minimum-size cycle cover $S' \subseteq S$ of $L_m$. 

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**Theorem** (F. Naserasr, Parreau, Valicov)

What is the complexity of LADDER CYCLE COVER?
MIN IDCODE for unit interval graphs

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**Theorem (F., Naserasr, Parreau, Valicov)**

MIN IDCODE for unit interval graphs of order $n$ can be reduced to LADDER CYCLE COVER for $L_{n+1}$ and an input of $n$ cycles.
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Complexity of MIN IDCODE for various graph classes

What is the decision complexity of MIN IDCODE here?

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