Bounding the identifying code number of a graph using its degree parameters
(a probabilistic approach)

Florent Foucaud (LaBRI)

GT probas, LaBRI - September 16th, 2011

joint work with Guillem Perarnau (UPC, Barcelona)
Let \( N[u] \) be the set of vertices \( v \) s.t. \( d(u, v) \leq 1 \)

**Definition** - Identifying code of \( G \) (Karpovsky, Chakrabarty, Levitin, 1998)

Subset \( C \) of \( V \) such that:

- \( C \) is a dominating set in \( G \): \( \forall u \in V, N[u] \cap C \neq \emptyset \), and
- \( C \) is a separating code in \( G \): \( \forall u \neq v \) of \( V \), \( N[u] \cap C \neq N[v] \cap C \)

Equivalently: \( (N[u] \Delta N[v]) \cap C \neq \emptyset \) (covering symmetric differences)

**Notation** - Identifying code number

\( \gamma_{ID}(G) \): minimum cardinality of an identifying code of \( G \)
Identifiable graphs

Let $N[u]$ be the set of vertices $v$ s.t. $d(u, v) \leq 1$

**Remark**

Not all graphs have an identifying code!

**Twins** = pair $u, v$ such that $N[u] = N[v]$.

A graph is identifiable iff it is twin-free (i.e. it has no twins).
Degree parameters of a graph

Graph $G = (V, E)$, vertex $v \in V$.

- **degree** of $v$: number of edges it is incident to
- **minimum degree** $\delta$ of $G$: min. degree of a vertex in $G$
- **maximum degree** $d$ of $G$: max. degree of a vertex in $G$
- **$d$-regular graph**: all vertices have degree $d$
Previous results

**Theorem** (Karpovsky, Chakrabarty, Levitin, 1998 + Gravier, Moncel, 2007)

Let $G$ be an identifiable graph with at least one edge, then

$$\lceil \log_2(n + 1) \rceil \leq \gamma^{ID}(G) \leq n - 1$$

**Theorem** (Karpovsky, Chakrabarty, Levitin, 1998)

Let $G$ be an identifiable graph with maximum degree $d$, then

$$\frac{2n}{d+2} \leq \gamma^{ID}(G)$$

**Conjecture** (F., Klasing, Kosowski, Raspaud, 2009+)

There exists a constant $c$, such that for every connected nontrivial identifiable graph $G$ of max. degree $d$,

$$\gamma^{ID}(G) \leq n - \frac{n}{d} + c$$

This would be tight. True for $d = 2$ and $d = n - 1$. 
The probabilistic method

Technique initiated, among others, by Pál Erdős
used mainly in combinatorics (Ramsey theory, graph theory, ...)

1. Define a suitable probability space
2. Select some object from this space using randomness
3. Prove that with nonzero probability, certain "good" conditions hold
4. Conclusion: there always exists a "good" object

Classic reference: Noga Alon and Joel Spencer, *The probabilistic method*
Corollaries

\( NF(G) \): proportion of non forced vertices of \( G \)

\[
NF(G) = \frac{\# \text{non-forced vertices in } G}{\# \text{vertices in } G}
\]

**Theorem (F., Perarnau, 2011+)**

There exists an integer \( d_0 \) such that for each identifiable graph \( G \) on \( n \) vertices having maximum degree \( d \geq d_0 \) and no isolated vertices,

\[
\gamma^{ID}(G) \leq n - \frac{n \cdot NF(G)^2}{85d}
\]

**Corollary**

- In general, \( NF(G) \geq \frac{1}{d+1} \) and \( \gamma^{ID}(G) \leq n - \frac{n}{\Theta(d^3)} \)
- If \( G \) is \( d \)-regular, \( NF(G) = 1 \) and \( \gamma^{ID}(G) \leq n - \frac{n}{85d} \).
- If \( G \) has clique number bounded by \( k \), \( NF(G) \geq \frac{1}{c(k)} \) and \( \gamma^{ID}(G) \leq n - \frac{n}{\Theta(d)} \).
Where are most of the $d$-regular graphs?

Let $G$ be a $d$-regular graph.

\[ \frac{2}{d} n \leq \frac{d-1}{d} n \]

\[ \gamma^{ID}(G) \geq \frac{2n}{d+2} \quad \text{Karpovsky et al. (1998)} \]

\[ \gamma^{ID}(G) \leq n - \frac{n}{d} + c \quad \text{Conjecture (2009)} \]
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Let $G$ be a random $d$-regular graph. Then a.a.s.

\[
\frac{2}{d} n \leq \gamma^{\text{ID}}(G) \leq \left(1 + o_d(1)\right) \frac{2 \log d}{d} n
\]

\[\frac{2 \log d}{d} n \leq (1 + o_d(1)) \frac{d-1}{d} n\]

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\frac{2}{d} n \leq \gamma_{ID}(G) \leq \frac{2 \log d}{d} n \\
\frac{d-1}{d} n \leq \gamma_{ID}(G) \leq \log d + \log \log d + O_d(1) n
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(1 + o_d(1)) \frac{\log d}{d} n \leq \gamma_{ID}(G) \leq (1 + o_d(1)) \frac{2 \log d}{d} n \\
\frac{\log d - 2 \log \log d}{d} n \leq \gamma_{ID}(G) \leq \frac{\log d + \log \log d + O_d(1)}{d} n
$$
Probability space $G^*_{n,d}$ of $d$-regular multigraphs on $n$ vertices.

- Take $nd$ vertices grouped in $n$ buckets of size $d$
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Probability space $G^*_{n,d}$ of $d$-regular (labelled) multigraphs on $n$ vertices.

**Proposition** (Bollobás, 1980 - Wormald, 1981)

Let $G \in G^*_{n,d}$. Then $Pr(G \text{ is simple}) \rightarrow e^{\frac{1-d^2}{4}} > 0$
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**Notation** - Simple random regular graphs

Let $\mathcal{G}_{n,d} = \mathcal{G}_{n,d}^* \mid$ the graph is simple.

$\mathcal{G}_{n,d}^*$: non-uniform distribution. $\mathcal{G}_{n,d} = \mathcal{G}$: uniform distribution
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**Theorem**: any property which holds a.a.s. for $\mathcal{G}_{n,d}^*$, also does for $\mathcal{G}_{n,d}$.
Remark

Other models are known, but do not provide a uniform distribution.
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\[ L_d(n) \text{: \# labelled } d\text{-regular graphs on } n \text{ vertices, } U_d(n) \text{: \# UNlabelled } d\text{-regular graphs.} \]

Bollobás, 1982: \[ U_d(n) \sim \frac{L_d(n)}{n} \sim \frac{(rn)!e^{(1-d^2)/4}}{(\frac{m}{2})!2^m/2(r!n!)} \]
(Note: no exact formula is known!)

Corollary: any property which holds a.a.s. for labelled \( d \)-regular graphs also does for unlabelled ones.
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**Proposition** (Bollobás, 1980 - Wormald, 1981)

\[ \mathbb{E}(\text{number of } k\text{-cycles in } G^*_{n,d}) \longrightarrow \frac{(d-1)^k}{2k} \]

In fact, stronger result: the distribution of the numbers of \( k \)-cycles for fixed \( k \in \{2, \ldots\} \) all jointly tend to independent Poisson variables of parameter \( \lambda_k = \frac{(d-1)^k}{2k} \).
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**Proposition**

Let \( G \in G_{n,d}^* \), then a.a.s. \( G \) is identifiable (no twins).
Graphs with girth at least 5

Proposition (F., Perarnau, 2011+)

Let $G$ be a twin-free graph on $n$ vertices having girth at least 5. Let $D$ be a 2-dominating set of $G$. If the subgraph induced by $D$, $G[D]$, has no isolated edge, $D$ is an identifying code of $G$. 

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**Proposition (F., Perarnau, 2011+)**

Let $G$ be a $d$-regular graph with girth at least 5. Then

$$\gamma_{ID}^D(G) \leq \frac{\log d + \log \log d + O_d(1)}{d} n$$
Sketch of the proof: construct 2-dominating set $D$

- $S \subseteq V$ at random, each element with probability $p$. 
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$$X_v = \begin{cases} 
0 & \text{if } |N[v] \cap S| \geq 2 \\
1 & \text{otherwise}
\end{cases}$$

$$Pr(X_v = 1) = (1 - p)^{d+1} + (d + 1)p(1 - p)^d \leq (1 + dp)e^{-dp}$$
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- $X(S) = \sum X_v$ (# non 2-dominated).

- $C = S \cup \{v : X_v = 1\}$, $p = \frac{\log d + \log \log d}{d}$

$$\mathbb{E}(|D|) = \mathbb{E}(|S|) + X(S) \leq \frac{\log d + \log \log d}{d}n + \frac{1 + \log d + \log \log d}{d \log d}n$$

$$\mathbb{E}(|D|) \leq \frac{\log d + \log \log d + O_d(1)}{d}n$$
Sketch of the proof: identifying code

\[
Y_{uv} = \begin{cases} 1 & \text{if} \ 0 \\ \text{otherwise} \end{cases}
\]

\[
\Pr(Y_{uv} = 1) \leq p^2 (1 - p)^2 d - 2 + (1 - p)^2 d + p (1 - p)^2 d - 1
\]

\[
\mathcal{C} = \mathcal{S} \cup \{v : X_v = 1\} \cup \{w : w \in N(u), Y_{uv} = 1\}
\]

\[
\mathbb{E}(|\mathcal{C}|) \leq \log d + \log \log d + O(d)
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Sketch of the proof: identifying code

\[ Y_{uv} = \begin{cases} 1 & \text{if} \ 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ \Pr(Y_{uv} = 1) \leq p^2 (1 - p)^2 d^{-2} + (1 - p)^2 d + p (1 - p)^2 d^{-1} \]

\[ C = S \cup \{ v : X_v = 1 \} \cup \{ w : w \in N(u), Y_{uv} = 1 \} \]

\[ E(|C|) \leq \log d + \log \log d \]

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Sketch of the proof: identifying code

\[ Y_{uv} = \begin{cases} 1 & \text{if} \, w \quad \text{SOLVED!} \\ 0 & \text{otherwise} \end{cases} \]

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\mathcal{C} = S \cup \{ v : X_v = 1 \} \cup \{ w : w \in N(u), \, Y_{uv} = 1 \}, \quad p = \frac{\log d + \log \log d}{d}
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\mathbb{E}(|\mathcal{C}|) \leq \frac{\log d + \log \log d + O_d(1)}{d} n
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Let $G$ be a random $d$-regular graph. Then a.a.s.

$$\gamma^{ID}(G) \leq \frac{\log d + \log \log d + O_d(1)}{d} n$$

Let $G$ be a $d$-regular graph of order $n$, taken u.a.r.: $G \in \mathcal{G}(n, d)$
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\Pr(G \text{ identifiable}) \xrightarrow{n \to \infty} 1
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Back to random regular graphs - upper bound

**Theorem (F., Perarnau, 2011+)**

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$$\mathbb{E}(C_3's) = e \frac{(d-1)^3}{6} \quad \mathbb{E}(C_4's) = e \frac{(d-1)^4}{8}$$
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$$\mathbb{E}(C_3 \text{'s}) = e \frac{(d-1)^3}{6} \quad \mathbb{E}(C_4 \text{'s}) = e \frac{(d-1)^4}{8}$$

$$\Pr(\#C_3 > \log \log n) \longrightarrow 0$$

$$\Pr(\#C_4 > \log \log n) \longrightarrow 0$$
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