Bounds on the size of identifying codes for graphs of maximum degree $\Delta$

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simple, undirected graph : models a building
simple detectors: able to detect a fire in a neighbouring room

goal: locate an eventual fire
Locating a fire in a building

simple detectors: able to detect a fire in a neighbouring room

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fire in room \( f \)
Locating a fire in a building

simple detectors: able to detect a fire in a neighbouring room

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fire in room $f$

the *identifying sets* of all vertices must be distinct
Identifying codes: definition

Definition: identifying code of a graph $G = (V, E)$ (Karpovskiy et al. 1998 [2])

subset $C$ of $V$ such that:

- $C$ is a dominating set in $G$, and
- for all distinct $u, v$ of $V$, $u$ and $v$ have distinct identifying sets: $N[u] \cap C \neq N[v] \cap C$
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Remark

Note: close to locating-dominating sets (Slater, Rall 84 [4])
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Notation

$\gamma_{id}(G)$: minimum cardinality of an identifying code in a graph $G$
Remark: not all graphs admit an identifying code

$u$ and $v$ are twin vertices if $N[u] = N[v]$.
A graph is identifiable iff it has no twin vertices.
**Identifiable graphs**

**Remark**: not all graphs admit an identifying code.

* u and v are *twin* vertices if $N[u] = N[v]$.

A graph is *identifiable* iff it has no twin vertices.

**Non-identifiable graphs**

[Diagram showing non-identifiable graph]
Remark: not all graphs admit an identifying code

\( u \) and \( v \) are twin vertices if \( N[u] = N[v] \).

A graph is identifiable iff it has no twin vertices.

Non-identifiable graphs

![Diagram of non-identifiable graphs](image-url)
Lower bound and maximum degree

Thm (Karpovski et al. 98 [2])

Let $G$ be an identifiable graph with $n$ vertices. Then

$$
\gamma_{id}(G) \geq \lceil \log_2(n + 1) \rceil.
$$
Lower bound and maximum degree

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**Characterization**

The graphs reaching this bound have been characterized (Moncel 06 [3]).
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Characterization

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Thm (Karpovski et al. 98 [2])

Let $G$ be an identifiable graph with $n$ vertices and maximum degree $\Delta$. Then

$$\gamma_{id}(G) \geq \frac{2n}{\Delta + 2}.$$
Graphs reaching the lower bound

Characterization

- $n$ vertices
- Independent set $C$ of size $\frac{2n}{\Delta + 2}$ (id. code)
- Every vertex of $C$ has exactly $\Delta$ neighbours
- $\frac{\Delta n}{\Delta + 2}$ vertices connected to exactly 2 code vertices each
Example: $D=$Petersen graph, $\Delta = 3$, $n = 10$
Graphs reaching the lower bound - example

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A general upper bound

**Thm (Gravier, Moncel 07 [1])**

Let $G$ be an identifiable connected graph with $n \geq 3$ vertices. Then $\gamma_{id}(G) \leq n - 1$. 

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Bounds on id codes

September 2009
A general upper bound

**Thm (Gravier, Moncel 07 [1])**

Let $G$ be an identifiable connected graph with $n \geq 3$ vertices. Then $\gamma_{id}(G) \leq n - 1$.

**Thm (Gravier, Moncel 07 [1])**

For all $n \geq 3$, there exist identifiable graphs with $n$ vertices with $\gamma_{id}(G) = n - 1$. 
Example: the star $K_{1,n-1}$
Upper bound - example

Example: the star $K_{1,n-1}$
Remark

All these graphs have a high maximum degree $\Delta(G) : n - 1$ or $n - 2$. 
Thm (F., Klasing, Kosowski and Raspaud 09)

Let $G$ be a connected identifiable graph of maximum degree $\Delta$. Then $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^4)}$.
If $G$ is regular, $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^2)}$. 
Result - general case

Thm (F., Klasing, Kosowski and Raspaud 09)

Let $G$ be a connected identifiable graph of maximum degree $\Delta$. Then $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^4)}$. If $G$ is regular, $\gamma_{id}(G) \leq n - \frac{n}{\Theta(\Delta^2)}$.

Sketch of the proof

- Greedily construct a 4-independant (resp. 2-independent) set $S$: distance between two vertices is at least 5 (resp. 3)
- take $C = V \setminus S$ as a code
- $C$ must be modified locally
Take any $\Delta$-regular graph $H$ with $m$ vertices
replace any vertex of $H$ by a clique of $\Delta$ vertices
Connected cliques

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Example: $H = K_4$
Connected cliques

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Exemple : $H = K_4$
Take any $\Delta$-regular graph $H$ with $m$ vertices

replace any vertex of $H$ by a clique of $\Delta$ vertices

Exemple : $H = K_4$

For every clique, at least $\Delta - 1$ vertices in the code

$\Rightarrow \gamma_{id}(G) \geq m \cdot (\Delta - 1) = n - \frac{n}{\Delta}$
Proposition

Let $K_{m,m}$ be the complete bipartite graph with $n = 2m$ vertices.

$id(K_{m,m}) = 2m - 2 = n - \frac{n}{\Delta}.$
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Let $K_{m,m}$ be the complete bipartite graph with $n = 2m$ vertices.

$$id(K_{m,m}) = 2m - 2 = n - \frac{n}{\Delta}.$$ 

Thm (Bertrand et al. 05)

Let $T^h_k$ be the $k$-ary tree with $h$ levels and $n$ vertices.

$$id(T^h_k) = \left\lceil \frac{k^2 n}{k^2 + k + 1} \right\rceil = n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}}.$$
**Thm (F., Klasing, Kosowski and Raspaud 09)**

Let $G$ be a connected triangle-free identifiable graph $G$ with $n \geq 3$ vertices and maximum degree $\Delta$. Then $\gamma_{id}(G) \leq n - \frac{n}{3\Delta + 3}$.

If $G$ is regular, $\gamma_{id}(G) \leq n - \frac{n}{2\Delta + 2}$. 
Thm (F., Klasing, Kosowski and Raspaud 09)

Let $G$ be a connected triangle-free identifiable graph $G$ with $n \geq 3$ vertices and maximum degree $\Delta$. Then $\gamma_{id}(G) \leq n - \frac{n}{3\Delta+3}$.
If $G$ is regular, $\gamma_{id}(G) \leq n - \frac{n}{2\Delta+2}$.

Sketch of the proof

- Greedily construct an independent set $S$ with special properties: $|S| \geq \frac{n}{\Delta+1}$
- Take $C = V \setminus S$ as a code
- Some vertices may not be identified correctly
- $\rightarrow$ locally modify $C$. It is possible to add not too much vertices to $C$
Thm (F., Klasing, Kosowski and Raspaud 09)

Let $G$ be an identifiable graph with $n$ vertices, of minimum degree $\delta \geq 2$ and girth $g \geq 5$.
Then $\gamma_{id}(G) \leq \frac{7n}{8} + 1$. 
Thm (F., Klasing, Kosowski and Raspaud 09)

Let $G$ be an identifiable graph with $n$ vertices, of minimum degree $\delta \geq 2$ and girth $g \geq 5$. Then $\gamma_{id}(G) \leq \frac{7n}{8} + 1$.

Sketch of the proof

- Construct a DFS spanning tree $T$ of $G$
- Partition the vertices into 4 classes $V_0, V_1, V_2, V_3$ depending on their level in $T$
- Take $C = V \setminus V_i$ as a code, $|V_i| \geq \frac{n}{4}$ : $|V_i| \leq \frac{3n}{4}$
- $C$ must be modified locally; the size of $C$ might increase
Graphs of girth at least 5

level 0
level 1
level 2
level 3
level 4
level 5
level 6
### Summary

<table>
<thead>
<tr>
<th></th>
<th>arbitrary graphs</th>
<th>$\Delta$-regular graphs</th>
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